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Influence of numerical discretizations on hitting probabilities for linear stochastic parabolic systems [☆,☆☆]

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ABSTRACT

This paper investigates the influence of standard numerical discretizations on hitting probabilities for a system of linear stochastic parabolic equations driven by space-time white noises. We establish lower and upper bounds for hitting probabilities of the associated numerical solutions of both temporal and spatial semi-discretizations in terms of Bessel–Riesz capacity and Hausdorff measure, respectively. Moreover, the critical dimensions of both temporal and spatial semi-discretizations turn out to be halves of those of the exact solution. This reveals that there exist some Borel sets A such that the probability of the event that the paths of the numerical solution hit A cannot converge to that of the exact solution when the stepsizes vanish.

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1. Introduction

Hitting probability is an active field of probability potential theory. Generally speaking, for an $\mathbb{R}^d$-valued random field $X = \{X(x), x \in \mathbb{R}^m\}$, its hitting probability concerns the lower and upper bounds of $\mathbb{P}\{X(I) \cap A \neq \emptyset\}$, where $I \subset \mathbb{R}^m$ is a fixed compact set with positive Lebesgue measure, and $A \subset \mathbb{R}^d$ is a Borel set. In this setting, a Borel set $A \subset \mathbb{R}^d$ is called *polar* for X if $\mathbb{P}\{X(I) \cap A \neq \emptyset\} = 0$; otherwise, A is called *nonpolar*. When X is the solution of a system of SPDEs, the hitting probability of X has been studied extensively (see e.g., [6–12,18,22–24]), which is usually bounded by the Bessel–Riesz capacity and Hausdorff measure of A (see [19, Appendix C] or subsection 2.3 for the definitions), namely,

$$c\,\mathrm{Cap}_{d-Q}(A) \le \mathbb{P}\left\{X(I) \cap A \neq \emptyset\right\} \le C\mathscr{H}_{d-Q}(A), \tag{1}$$

for some $Q > 0$ and $c, C > 0$. It is well known that the *critical dimension* Q (see e.g., [9,13]) is an important parameter that is highly related to the polarity of a Borel set A. When X is approximated by a perturbation, particularly by a numerical solution provided that X is the exact solution of a system of SPDEs, a natural question is what are the influence of the perturbation on critical dimensions of hitting probabilities. To the best of our knowledge, there is no result on this problem.

In this paper, we investigate the influence on hitting probabilities of numerical discretizations for the following system of linear stochastic parabolic equations

$$\begin{cases} \partial_t u^j(t,x) - \partial_{xx} u^j(t,x) = \dot{W}^j(t,x), \\ u^j(t,0) = u^j(t,1) = 0,\ t \ge 0, \\ u^j(0,x) = 0,\ x \in [0,1], \end{cases} \tag{2}$$

for $j = 1, \ldots, d$, where $u(t,x) = (u^1(t,x), \ldots, u^N(t,x))$, and $\{W^k\}_{k=1,\ldots,d}$ are d independent Brownian sheets on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$. The hitting probabilities of the exact solution for system (2) are well established by [1,7], which suggest that the critical dimensions in time and space directions are respectively 4 and 2, that is for any bounded Borel set A in $\mathbb{R}^d$,

$$\begin{aligned} &c\,\mathrm{Cap}_{d-4}(A) \le \mathbb{P}\left\{u([T_0,T] \times \{x\}) \cap A \neq \emptyset\right\} \le C\mathscr{H}_{d-4}(A), \\ &c\,\mathrm{Cap}_{d-2}(A) \le \mathbb{P}\left\{u(\{t\} \times [\epsilon, 1-\epsilon]) \cap A \neq \emptyset\right\} \le C\mathscr{H}_{d-2}(A). \end{aligned}$$

Here and after, $T_0 \in (0,T)$, $\epsilon \in (0, \frac{1}{2})$ are fixed numbers, and c, C are generic positive constants that may differ from one place to another. Numerical approximations of stochastic parabolic equations have been studied extensively, which converge to exact solutions in some sense (see e.g., [3,4,17]), but may reflect new and interesting properties, see [5] for the effect of the length of the time-steps on the quadratic and quartic variations, see [2] for the influence of regularity of the test function on weak convergence of numerical solutions. The main result of this paper reveals that the critical dimensions of both temporal and spatial semi-discretizations considered in Section 2 are halves of those of the exact solution. This indicates that there exist some Borel sets A such that the probability of the event that the paths of the numerical solution hit A cannot converge to that of the exact solution. This property may be linked to the regularity of trajectories (see e.g. [5]).

For the spatial semi-discretization of system (2), we introduce the finite difference method (FDM) and the spectral Galerkin method (SGM), and formulate the corresponding numerical solutions $u^{j,N}(t,x)$ as stochastic integrals associated with discrete Green functions. For FDM, the associated numerical solution $\{u^{j,N}(t,x); x \in (x_i, x_{i+1}]\}$ is the linear interpolation of $u^{j,N}(t,x_i)$ and $u^{j,N}(t,x_{i+1})$. We find that for any fixed space grid point $x_i = i/N \in [\epsilon, 1-\epsilon]$, $U^N(t,x_i) = \left(u^{1,N}(t,x_i), \ldots, u^{d,N}(t,x_i)\right)$ given by FDM is Hölder continuous with respect to $t \in [T_0,T]$ with the optimal Hölder exponent $\frac{1}{2}$, which is crucial to conclude that the critical dimension associated to the time direction of FDM is 2. More precisely, for any bounded Borel set A in $\mathbb{R}^d$,

$$c\,\mathrm{Cap}_{d-2}(A) \le \mathbb{P}\left\{U^N([T_0,T] \times \{x_i\}) \cap A \neq \emptyset\right\} \le C\mathscr{H}_{d-2}(A),$$

which can also be extended to the case of SGM. By noticing that $U^N(t,x)$ based on SGM is still a two-variable random field indexed by $(t,x)\in[0,\infty)\times[0,1]$, we further investigate its critical dimension in space direction. The main difficulty lies in establishing lower bounds for the Hölder continuity and conditional variance in terms of the associated canonical metric $|x-y|$, which is overcome by refined estimates of the discrete Green function. These yield that the critical dimension associated to the space direction of SGM is 1, i.e., for any bounded Borel set A in $\mathbb{R}^d$,

$$c\,\mathrm{Cap}_{d-1}(A)\le \mathbb{P}\left\{U^N(\{t\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\}\le C\mathscr{H}_{d-1}(A).$$

This property may not be extended to the numerical approximation given by FDM whose trajectories $[0,1]\ni x\mapsto U^N(t,x)$ are piecewise linear, since it is a challenge to obtain the lower bound of conditional variance in terms of the associated canonical metric $|x-y|$.

For the temporal semi-discretization of system (2), we apply the exponential Euler method (EEM) with time stepsize $1/M$, $M\in\mathbb{N}_+$. For any fixed time grid point $t_j=\frac{j}{M}\in[T_0,T]$, the numerical solution $U_M(t_j,\cdot)$ of EEM is smoother than the exact solution since the temporal discretization avoids the treatment of the singularity of the Green function $G_t(x,y)$ near $t=0$. Actually, making use of this property, we show that $U_M(t_j,x)$ is Lipschitz continuous with respect to $x\in[\epsilon,1-\epsilon]$, and 1 is exactly the optimal Hölder exponent. As a consequence, the critical dimension associated to the space direction of EEM is also 1:

$$c\,\mathrm{Cap}_{d-1}(A)\le \mathbb{P}\left\{U_M(\{t_j\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\}\le C\mathscr{H}_{d-1}(A),$$

for any bounded Borel set A in $\mathbb{R}^d$. For the continuous EEM numerical solution $U_M(t,x)$, we can only obtain the upper bound of hitting probabilities in time direction in terms of Hausdorff measure since $U_M(t,x)$ is smoother in every subinterval (t_i,t_{i+1}) than in grid points. It is worth mentioning that different from the infinite dimensional case, the continuous EEM numerical solution for the system of finite dimensional Ornstein–Uhlenbeck equations preserves the critical dimension of the exact solution of the original system.

The rest of this paper is organized as follows. Section 2 states some preliminaries, including the model, numerical discretizations, and hitting probabilities of the exact solution. Our main results on the upper and lower bounds for hitting probabilities of semi-discretizations are presented in subsection 2.4. Detailed proofs of main results are postponed to Section 3. Section 4 compares the continuous version of the time discretization for system (2) and that of a system of finite dimensional Ornstein–Uhlenbeck equations.

2. Preliminaries and main results

In this section, we introduce the model and its numerical discretizations, and present main results on hitting probabilities of numerical discretizations.

2.1. The model

Consider the following linear stochastic parabolic equation:

$$\begin{cases}\partial_t v(t,x)-\partial_{xx}v(t,x)=\dot{W}(t,x),\\ v(t,0)=v(t,1)=0,\ t\ge 0,\\ v(0,x)=0,\ x\in[0,1],\end{cases}\tag{3}$$

where W is a Brownian sheet on $[0,\infty)\times[0,1]$. Then

$$v(t,x)=\int_0^t\int_0^1 G_{t-r}(x,z)W(\mathrm{d}r,\mathrm{d}z),\ (t,x)\in[0,\infty)\times[0,1],$$

and the components u^j, $j=1,\dots,d$, in (2) are independent copies of v. Here, the Green function G has the expression

$$G_t(x,y)=\sum_{k=1}^{\infty}e^{-\pi^2k^2t}e_k(x)e_k(y) \tag{4}$$

with $e_k(x)=\sqrt{2}\sin(k\pi x)$, $k\geq 1$. We recall another equivalent formulation

$$G_t(x,y)=\frac{1}{\sqrt{4\pi t}}\sum_{n=-\infty}^{+\infty}\left(e^{-\frac{(x-y-2n)^2}{4t}}-e^{-\frac{(x+y-2n)^2}{4t}}\right).$$

Let $P_t(x,y)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$ be the heat kernel on $\mathbb{R}$. We remark that

$$G_t(x,y)\leq P_t(x,y), \tag{5}$$

and for every $0<\epsilon<\frac{1}{2}$,

$$G_t(x,y)\geq(1-2e^{-\frac{\epsilon^2}{t}})P_t(x,y),\ x,y\in[\epsilon,1-\epsilon] \tag{6}$$

(see e.g. [16, Lemmas 3.1 and 3.3] for their proofs). We also need the following lower bound of the Green function, whose proof is postponed to Appendix A.1.

Lemma 2.1. *For any $\epsilon\in(0,\frac{1}{2})$, there exists $C:=C(\epsilon,T)>0$ such that*

$$G_t(x,x)\geq Ct^{-\frac{1}{2}},\quad \forall\, x\in[\epsilon,1-\epsilon],\ t\in(0,T].$$

2.2. Numerical methods

In this part, we introduce some spatial and temporal discretizations for the linear stochastic parabolic system (2).

2.2.1. Spatial discretizations

For spatial discretizations of (2), we introduce the finite difference method and the spectral Galerkin method. Their numerical solutions can be written as

$$v^N(t,x)=\int_0^t\int_0^1 G^N_{t-r}(x,z)W(\mathrm{d}r,\mathrm{d}z),\ (t,x)\in(0,\infty)\times[0,1], \tag{7}$$

where G^N is given in (10) for FDM and in (12) for SGM. The associated numerical solution to (2) is denoted by

$$U^N(t,x)=\big(u^{1,N}(t,x),\dots,u^{d,N}(t,x)\big), \tag{8}$$

where $u^{j,N}(t,x)$ are generated via replacing W in (7) by W^j, $j=1,\dots,d$. For $N\geq 2$, denote $\mathbb{Z}_N:=\{1,\dots,N-1\}$.

Finite difference method: Using the central difference, the finite difference method of (3) is proposed in [17]. For every integer $N\geq 2$, the associated numerical solution $v^N(t,x)$ is constructed as follows. Let $\{v^N(t,\frac{k}{N})\}_{k\in\mathbb{Z}_N}$ solve

$$\begin{aligned}\mathrm{d}v^N(t,\frac{k}{N})=&N^2\left(v^N(t,\frac{k+1}{N})-2v^N(t,\frac{k}{N})+v^N(t,\frac{k-1}{N})\right)\mathrm{d}t\\&+N\mathrm{d}\left(W(t,\frac{k+1}{N})-W(t,\frac{k}{N})\right),\end{aligned}$$

$$v^N(0, \frac{k}{N}) = 0, \quad v^N(t,0) = v^N(t,1) = 0. \tag{9}$$

For $x \in [\frac{k}{N}, \frac{k+1}{N})$, $v^N(t,x)$ is defined by the polygonal approximation

$$v^N(t,x) := v^N(t, \frac{k}{N}) + (Nx - k)\left(v^N(t, \frac{k+1}{N}) - v^N(t, \frac{k}{N})\right).$$

By introducing $v_k^N(t) := v^N(t, \frac{k}{N})$ and $B_k^N(t) := \sqrt{N}\left(W(t, \frac{k+1}{N}) - W(t, \frac{k}{N})\right)$, we rewrite (9) as

$$\mathrm{d}v_k^N(t) = N^2 \sum_{i=1}^{N-1} D_{ki} v_i^N(t)\mathrm{d}t + \sqrt{N}\mathrm{d}B_k^N(t),$$
$$v_k^N(0) = 0, \quad k = 1, \dots, N-1,$$

where $D = (D_{ki})$ is a square matrix of size $N-1$ with $D_{kk} = -2$, $D_{ki} = 1$ for $|k-i| = 1$, and $D_{ki} = 0$ for $|k-i| > 1$. The vectors $f_1, \dots f_{N-1}$ defined by $f_j = (f_j(1), \dots, f_j(N-1))^\top$ with

$$f_j(k) = \sqrt{\frac{2}{N}} \sin\left(j\frac{k}{N}\pi\right), \quad k = 1, \dots, N-1$$

form an orthonormal basis of $\mathbb{R}^{N-1}$, which are also eigenvectors of $N^2 D$ corresponding to the eigenvalues $\{-4N^2 \sin^2\left(\frac{j}{2N}\pi\right),\ j = 1, \dots, N-1\}$. By the variation of constant formula, the solution of (9) is

$$v^N(t,x_i) = \int_0^t \int_0^1 \sum_{k=1}^{N-1} \exp\left(-4N^2 \sin^2\left(\frac{k}{2N}\pi\right)(t-r)\right) e_k(x_i) e_k(\kappa_N(z)) W(\mathrm{d}r, \mathrm{d}z),$$

where $x_i := \frac{i}{N}$, $t > 0$, and $\kappa_N(z) := \frac{[Nz]}{N}$ with $[\cdot]$ being the floor function (see e.g. [17, formula (2.7)] with $f \equiv 0$ and $\sigma \equiv 1$). We remark that $-k^2\pi^2 \leq -4N^2 \sin^2\left(\frac{k}{2N}\pi\right) \leq -4k^2$, $\forall k \in \mathbb{Z}_N$ and $N \geq 2$. By the polygonal interpolation, we obtain the continuous numerical solution (7) of FDM, where the associated discrete Green function is

$$G_t^N(x,y) = \sum_{k=1}^{N-1} \exp\left(-4N^2 \sin^2\left(\frac{k}{2N}\pi\right)t\right) e_k^N(x) e_k(\kappa_N(y)) \tag{10}$$

with

$$e_k^N(x) := e_k(\kappa_N(x)) + N(x - \kappa_N(x))\left[e_k\left(\kappa_N(x) + \frac{1}{N}\right) - e_k(\kappa_N(x))\right],\ x \in [0,1]. \tag{11}$$

Spectral Galerkin method: The spectral Galerkin approximation for (3) is studied in [20] by rewriting (3) as an infinite dimensional stochastic evolution equation

$$\mathrm{d}v(t) = \Delta v(t)\mathrm{d}t + \mathrm{d}V_t,\ v(0) = 0,$$

where Δ is the Dirichlet Laplacian, and $V_t = \sum_{k=1}^\infty \beta_k(t) e_k$ is some cylindrical Wiener process on $H := L^2(0,1)$ with $\{\beta_k(t), t \geq 0\}_{k \geq 1}$ being a sequence of independent Brownian motions on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$. Define the finite dimensional subspace $H_N := \mathrm{span}\{e_k, k \in \mathbb{Z}_N\}$ of H, and the projection operator $P_N : H \to H_N$ by $P_N h = \sum_{k=1}^{N-1} \langle e_k, h\rangle_H e_k$, $h \in H$. Then by [20, subsection 3.1], the spectral Galerkin approximation for (3) in H_N is to find an $\{\mathscr{F}_t\}$-adapted H_N-valued process $v^N = \{v_t^N, t > 0\}$ such that

$$\mathrm{d}v_t^N = P_N \Delta v_t^N \mathrm{d}t + \sum_{k=1}^{N-1} e_k \mathrm{d}\beta_k(t),\ v_0^N = 0,$$

which is equivalent to a system of stochastic differential equations:

$$\mathrm{d}\langle v_t^N, e_k\rangle_H = -k^2\pi^2\langle v_t^N, e_k\rangle_H \mathrm{d}t + \mathrm{d}\beta_k(t),\ \langle v_0^N, e_k\rangle_H = 0,\ k \in \mathbb{Z}_N.$$

Noticing $\sum_{k=1}^{N-1}\langle v_t^N, e_k\rangle_H e_k(x) = v^N(t,x)$ in distribution sense, one can verify that

$$v^N(t,x) = \int_0^t\int_0^1 G_{t-r}^N(x,z)W(\mathrm{d}r,\mathrm{d}z),\ (t,x) \in (0,\infty)\times[0,1],$$

where the discrete heat kernel associated to SGM is

$$G_t^N(x,y) = \sum_{k=1}^{N-1}\exp(-k^2\pi^2 t)e_k(x)e_k(y). \tag{12}$$

2.2.2. Temporal discretization

Introducing a time stepsize $\delta t = 1/M$, $M \in \mathbb{N}_+$, the numerical solution of the exponential Euler method for (3) is given by $v_M(t_0,x) = 0$,

$$v_M(t_i,x) = \int_0^1 G_{\delta t}(x,z)v_M(t_{i-1},z)\mathrm{d}z + \int_{t_{i-1}}^{t_i}\int_0^1 G_{\delta t}(x,z)W(\mathrm{d}r,\mathrm{d}z),$$

where $t_i = i\delta t$, $i \in \mathbb{N}_+$. Hence, for $i \in \mathbb{N}_+$,

$$v_M(t_i,x) = \int_0^{t_i}\int_0^1 G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z)W(\mathrm{d}r,\mathrm{d}z). \tag{13}$$

The associated numerical solution of system (2) is denoted by

$$U_M(t_i,x) = \left(u_M^1(t_i,x),\ldots,u_M^d(t_i,x)\right),\ i \in \mathbb{N}_+, \tag{14}$$

where $u_M^j(t_i,x)$ is generated by replacing W in (13) by W^j for $j \in \{1,\ldots,d\}$.

2.3. Hitting probability

Given two random variables X and Y, we denote $\mathrm{Var}X := \mathbb{E}|X-\mathbb{E}X|^2$, $\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)]$, and $\mathrm{Corr}(X,Y) := \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}}$. For any Borel set $F \subset \mathbb{R}^d$, define $\mathcal{P}(F)$ to be the set of all probability measures with compact support in F. For $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\beta \in \mathbb{R}$, let $I_\beta(\mu)$ denote the β-dimensional energy of μ, i.e.,

$$I_\beta(\mu) := \iint \mathrm{K}_\beta(\|x-y\|)\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$

where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$, and

$$\mathrm{K}_\beta(r) := \begin{cases} r^{-\beta}, & \text{if } \beta > 0, \\ \log\left(\frac{e}{r\wedge 1}\right), & \text{if } \beta = 0, \\ 1, & \text{if } \beta < 0. \end{cases}$$

For any $\beta \in \mathbb{R}$ and a Borel set $F \subset \mathbb{R}^d$, $\mathrm{Cap}_\beta(F)$ denotes the β-dimensional Bessel–Riesz capacity of F, that is,

$$\mathrm{Cap}_\beta(F) := \left[\inf_{\mu\in\mathcal{P}(F)} I_\beta(\mu)\right]^{-1},$$

where $1/\infty := 0$. Given $\beta \geq 0$, the β-dimensional Hausdorff measure of F is defined by

$$\mathscr{H}_\beta(F) = \lim_{\varepsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : F \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \sup_{i \geq 1} r_i \leq \varepsilon \right\},$$

where $B(x,r)$ denotes the open Euclidean ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. When $\beta < 0$, $\mathscr{H}_\beta(F)$ is defined to be infinite. We refer to [19, Appendix C] for more details about the Bessel–Riesz capacity and the Hausdorff measure.

Based on the Bessel–Riesz capacity and Hausdorff measure, we present the lower and upper bounds for hitting probabilities of the exact solution of (2), which is shown in [7, Theorem 4.6 & Remark 4.7] (see also [1, Section 4]).

Proposition 2.2. *[7] Let $u(t,x) = (u^1(t,x), \ldots, u^d(t,x))$ be the exact solution of (2). Fix $L > 0$ and $0 < \epsilon \ll 1$. Then there exist $C_i = C_i(T_0, T, d, \epsilon, L)$, $i = 1, \ldots, 6$, such that for any Borel set $A \subset [-L, L]^d$,*

$$C_1 \operatorname{Cap}_{d-6}(A) \leq \mathbb{P}\{u([T_0,T] \times [\epsilon, 1-\epsilon]) \cap A \neq \emptyset\} \leq C_2 \mathscr{H}_{d-6}(A),$$
$$C_3 \operatorname{Cap}_{d-2}(A) \leq \mathbb{P}\{u(\{t\} \times [\epsilon, 1-\epsilon]) \cap A \neq \emptyset\} \leq C_4 \mathscr{H}_{d-2}(A),$$
$$C_5 \operatorname{Cap}_{d-4}(A) \leq \mathbb{P}\{u([T_0,T] \times \{x\}) \cap A \neq \emptyset\} \leq C_6 \mathscr{H}_{d-4}(A),$$

where $t \in [T_0, T]$ and $x \in [\epsilon, 1-\epsilon]$.

2.4. Main results

The inequality (1) gives lots of information about hitting probabilities for the case of $d \neq Q$. In particular, if a random field X satisfies (1) with the critical dimension Q, then points are polar for X when $d > Q$, and are nonpolar for X when $d < Q$. In this part, we state our main results of this paper in Theorems 2.3, 2.4, and 2.5 for FDM, SGM, and EEM, respectively, which can be summarized as follows: for a fixed stepsize, the critical dimensions of both temporal and spatial semi-discretizations are halves of those of the exact solution. Since the stepsize is specified in advance, the generic constants C_i in Theorems 2.3, 2.4, and 2.5 are allowed to depend on the partition parameter N or M. We first give the hitting probabilities of the spatial semi-discretization of FDM in time direction.

Theorem 2.3. *Let $0 < \epsilon < \frac{1}{2}$ and $U^N(t,x)$ defined in (8) be the numerical solution of FDM for system (2). Fix $L > 0$. Then for sufficiently large N, there exist $C_i = C_i(N, L, \epsilon, T_0, T, d)$, $i = 1, 2$, such that for any Borel set $A \subset [-L, L]^d$,*

$$C_1 \operatorname{Cap}_{d-2}(A) \leq \mathbb{P}\left\{U^N([T_0,T] \times \{x\}) \cap A \neq \emptyset\right\} \leq C_2 \mathscr{H}_{d-2}(A),$$

where $x \in [\epsilon, 1-\epsilon]$.

Theorem 2.3 can be extended to the case of SGM. In addition, the numerical solution based on SGM is still a continuous Gaussian random field indexed by $(t,x) \in [0,\infty) \times [0,1]$, hence we further investigate its hitting probabilities in space direction.

Theorem 2.4. *Let $0 < \epsilon < \frac{1}{2}$ and $U^N(t,x)$ defined in (8) be the numerical solution of SGM for system (2). Fix $L > 0$. Then for sufficiently large N, there exist $C_i = C_i(N, L, \epsilon, T_0, T, d)$, $i = 1, 2, 3, 4$, such that for all Borel set $A \subset [-L, L]^d$,*

$$C_1 \operatorname{Cap}_{d-1}(A) \leq \mathbb{P}\left\{U^N(\{t\} \times [\epsilon, 1-\epsilon]) \cap A \neq \emptyset\right\} \leq C_2 \mathscr{H}_{d-1}(A), \tag{15}$$

$$C_3 \operatorname{Cap}_{d-2}(A) \leq \mathbb{P}\left\{U^N([T_0,T] \times \{x\}) \cap A \neq \emptyset\right\} \leq C_4 \mathscr{H}_{d-2}(A), \tag{16}$$

where $x \in [\epsilon, 1-\epsilon]$ and $t \in [T_0, T]$.

In (15) and (16), the constants C_i depend on the partition parameter N. Consequently, when letting N tend to ∞, we cannot obtain

$$C_1\,\mathrm{Cap}_{d-1}(A) \le \mathbb{P}\left\{U^{\infty}(\{t\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\} \le C_2\mathscr{H}_{d-1}(A),$$
$$C_3\,\mathrm{Cap}_{d-2}(A) \le \mathbb{P}\left\{U^{\infty}([T_0,T]\times\{x\})\cap A\neq\emptyset\right\} \le C_4\mathscr{H}_{d-2}(A),$$

where $U^{\infty} := \lim\limits_{N\to\infty} U^N$ in some sense is a formal notation. This reveals the differences between the cases $N=\infty$ and $N<\infty$. A similar property also holds for the temporal semi-discretization. For the temporal semi-discretization based on EEM, we study its hitting probabilities in space direction.

Theorem 2.5. *Let $0<\epsilon<\frac{1}{2}$ and $U_M(t,x)$ defined in (14) be the numerical solution of EEM for system (2). Fix $L>0$. Then for $M\ge 3$, there exist $C_i=C_i(M,L,\epsilon,T_0,T,d)$, $i=1,2$, such that for any Borel set $A\subset[-L,L]^d$,*

$$C_1\,\mathrm{Cap}_{d-1}(A) \le \mathbb{P}\left\{U_M(\{t\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\} \le C_2\mathscr{H}_{d-1}(A),$$

where $t\in\{\frac{1}{M},\frac{2}{M},\dots,\}\cap[T_0,T]$.

Theorems 2.3, 2.4 and 2.5 reveal that for some Borel sets A, the probability of the event that paths of the numerical solution hit A cannot converge to that of the exact solution. More precisely, by Frostman's theorem ([19, Appendix C, Theorem 2.2.1]), for any compact set $A\subset\mathbb{R}^d$,

$$\begin{aligned}\dim_{\mathrm{H}}(A) &= \sup\{s>0:\mathscr{H}_s(A)=\infty\} = \inf\{s>0:\mathscr{H}_s(A)=0\}\\ &= \sup\left\{s>0:\mathrm{Cap}_s(A)>0\right\} = \inf\left\{s>0:\mathrm{Cap}_s(A)=0\right\},\end{aligned}$$

where $\dim_{\mathrm{H}}(A)$ is the Hausdorff dimension of A. Therefore,

$$\mathrm{Cap}_{d-Q_{\mathrm{exact}}}(A)>0, \quad \forall\, d<\dim_{\mathrm{H}}(A)+Q_{\mathrm{exact}},$$

and

$$\mathscr{H}_{d-\frac{1}{2}Q_{\mathrm{exact}}}(A)=0, \quad \forall\, d>\dim_{\mathrm{H}}(A)+\frac{1}{2}Q_{\mathrm{exact}}.$$

Let a random field X satisfy (1) with $Q=Q_{\mathrm{exact}}$, and a sequence of random fields $\{X^n\}_{n\ge n_0}$ with some $n_0\in\mathbb{N}$ satisfy (1) with $Q=\frac{1}{2}Q_{\mathrm{exact}}$. Then for any bounded Borel set $A\subset\mathbb{R}^d$ with $\dim_{\mathrm{H}}(A)\in(d-Q_{\mathrm{exact}}, d-\frac{1}{2}Q_{\mathrm{exact}})$, it holds that

$$\mathbb{P}\left\{X^n(I)\cap A\neq\emptyset\right\} \le C\mathscr{H}_{d-\frac{1}{2}Q_{\mathrm{exact}}}(A)=0, \quad \forall\, n\ge n_0,$$
$$\mathbb{P}\left\{X(I)\cap A\neq\emptyset\right\} \ge C\,\mathrm{Cap}_{d-Q_{\mathrm{exact}}}(A)>0,$$

which indicates that

$$\lim_{n\to\infty}\mathbb{P}\left\{X^n(I)\cap A\neq\emptyset\right\}=0<\mathbb{P}\left\{X(I)\cap A\neq\emptyset\right\}. \tag{17}$$

Proposition 2.2 shows that for the exact solution u of system (2), the critical dimension $Q^t_{\mathrm{exact}}=4$ in time direction and the critical dimension $Q^x_{\mathrm{exact}}=2$ in space direction. Taking $X=u$ and $X^n=U^n$ (or $X^n=U_n$) in (17), the following results follow from Theorems 2.3, 2.4, and 2.5.

Corollary 2.6. *For any bounded Borel set $A\subset\mathbb{R}^d$ with $\dim_{\mathrm{H}}(A)\in(d-4,d-2)$,*

$$\lim_{N\to\infty}\mathbb{P}\left\{U^N([T_0,T]\times\{x\})\cap A\neq\emptyset\right\}=0<\mathbb{P}\left\{u([T_0,T]\times\{x\})\cap A\neq\emptyset\right\},$$

and for any bounded Borel set $A\subset\mathbb{R}^d$ with $\dim_{\mathrm{H}}(A)\in(d-2,d-1)$,

$$\lim_{M\to\infty}\mathbb{P}\left\{U_M(\{t\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\}=0<\mathbb{P}\left\{u(\{t\}\times[\epsilon,1-\epsilon])\cap A\neq\emptyset\right\}.$$

For example, for $d=3$ and each $y\in\mathbb{R}^3$, $\dim_{\mathrm{H}}(\{y\})=0$ (see e.g., [15, Example 2.2]), and hence $3=d\in(\dim_{\mathrm{H}}(\{y\})+2,\dim_{\mathrm{H}}(\{y\})+4)=(2,4)$. Thus, for fixed $x\in[\epsilon,1-\epsilon]$, all points $y\in\mathbb{R}^3$ are nonpolar for $u(\cdot,x)$ but polar for the spatial semi-discretization $U^N(\cdot,x)$. From Corollary 2.6, we conclude that there exist some Borel sets A such that the probability of the event that the paths of the numerical solution hit A cannot converge to that of the exact solution when the stepsizes vanish.

3. Proofs of main results

In this section, we present the proofs of main results in subsection 2.4. We first present in Proposition 3.1 the criterion about hitting probabilities of a general Gaussian random field, which is a reformulation of [1, Theorem 2.1] and [9, Theorems 2.1 & 2.6], and will be applied to prove the hitting probabilities for the numerical discretizations. For a set $I\subset\mathbb{R}^m$, we denote $I^\delta:=\cup_{y\in I}\{x\in\mathbb{R}^m:\|x-y\|\le\delta\}$ the δ-neighborhood of I.

Proposition 3.1. *Let $I=[a,b]:=\prod_{j=1}^m[a_j,b_j]$ $(a_j<b_j)$ be an interval or a rectangle in $\mathbb{R}^m$ and $X=\{X(x),\,x\in\mathbb{R}^m\}$ be an $\mathbb{R}^d$-valued Gaussian random field with coordinate processes $X_1,\dots,X_d$ being independent copies of a real-valued, centered Gaussian random field $X_0=\{X_0(x),\,x\in\mathbb{R}^m\}$. Assume that the following conditions* (C0), (C1) *and* (C2) *(or* (C2)$'$*) hold:*

(C0) *for some $\delta>0$, $\mathbb{E}|X_0(x)|^2\ge c_1$ holds for any $x\in I^\delta$;*
(C1) *there exists $H=(H_1,\dots,H_m)\in(0,1]^m$ such that for all $x,y\in I$,*

$$c_2\sum_{j=1}^m\left|x_j-y_j\right|^{2H_j}\le\mathbb{E}\left|X_0(x)-X_0(y)\right|^2\le c_3\sum_{j=1}^m\left|x_j-y_j\right|^{2H_j};$$

(C2) *$H\in(0,1)^m$ and there exist $c_4,\eta>0$ such that for any $x,y\in I$,*

$$\left|\mathbb{E}|X_0(x)|^2-\mathbb{E}|X_0(y)|^2\right|\le c_4\sum_{j=1}^m\left|x_j-y_j\right|^{H_j(1+\eta)},\tag{18}$$

$$\mathrm{Corr}(X_0(x),X_0(y))<1,\quad\forall\,x\ne y;\tag{19}$$

(C2)$'$ *$H=(1,\dots,1)$ and there exists positive constant c_4 such that for all $x,\ y\in I$,*

$$\mathrm{Var}\left(X_0(x)|X_0(y)\right)\ge c_4\left\|x-y\right\|^2.$$

Here, $c_i,\ i=1,2,3,4$, are independent of x, y, and $\mathrm{Var}\left(X_0(x)|X_0(y)\right)$ denotes the conditional variance of $X_0(x)$ given $X_0(y)$. Then for fixed $L>0$, there exist positive constants c_5, c_6 such that for every Borel set $A\subset[-L,L]^d$,

$$c_5\,\mathrm{Cap}_{d-Q}(A)\le\mathbb{P}\left\{X(I)\cap A\ne\emptyset\right\}\le c_6\mathscr{H}_{d-Q}(A),\tag{20}$$

where $Q:=\sum_{j=1}^m 1/H_j$.

For the sake of completeness, we give the proof of Proposition 3.1 in Appendix A.2. We remark that the right side of (C1) implies that there exists $c_7>0$ such that

$$\mathbb{E}|X_0(x)|^2\le 2c_3\sum_{j=1}^m\left|x_j\right|^{2H_j}+2\mathbb{E}|X_0(0)|^2\le c_7,\quad\forall\,x\in I.\tag{21}$$

We would like to mention that for Gaussian random fields, (C0) and the upper bound in (C1) suffice to derive the upper bound in (20), and the lower bounds in (C1) and (C2) (or (C2)$'$) are used to deduce the lower bound in (20). For the case of $H=(1,\dots,1)\in\mathbb{R}^m$, the following facts are helpful for estimating the conditional variance in (C2)$'$:

If (Y, Z) is a centered Gaussian vector, then

$$\mathrm{Var}(Y|Z) = \frac{\left(\mathbb{E}|Y-Z|^2 - \left(\sqrt{\mathbb{E}|Y|^2} - \sqrt{\mathbb{E}|Z|^2}\right)^2\right)\left(\left(\sqrt{\mathbb{E}|Y|^2} + \sqrt{\mathbb{E}|Z|^2}\right)^2 - \mathbb{E}|Y-Z|^2\right)}{4\mathbb{E}|Z|^2}. \tag{22}$$

By [14, Proposition 3.13], (22) has another equivalent formulation:

$$\mathrm{Var}\,(Y|Z) = \frac{\mathrm{Var}\,Y \mathrm{Var}\,Z - \mathrm{Cov}(Y,Z)^2}{\mathrm{Var}\,Z}. \tag{23}$$

3.1. *Proof of Theorem 2.3*

Based on Proposition 3.1, we only need to prove that the numerical solution v^N of the spatial discretization of FDM satisfies (C0)-(C2). The following lemma is instrumental for deriving the optimal Hölder continuity of $v^N(t,x)$ with respect to $x \in [\epsilon, 1-\epsilon]$.

Lemma 3.2. *Let $0 < \epsilon < \frac{1}{2}$ and $N > 14$. Then for any $x, y \in [\epsilon, 1-\epsilon]$,*

$$|e_1(x) - e_1(y)|^2 + |e_2(x) - e_2(y)|^2 \geq c(\epsilon)|x-y|^2, \tag{24}$$

$$|e_1^N(x) - e_1^N(y)|^2 + |e_2^N(x) - e_2^N(y)|^2 \geq c(\epsilon, N)|x-y|^2, \tag{25}$$

where $c(\epsilon)$ and $c(\epsilon, N)$ are positive constants.

Proof. Without loss of generality, we always assume $\epsilon \leq x < y \leq 1-\epsilon$.

The proof of (24) is divided into two cases.

Case 1: $x + y \in [2\epsilon, 1-\epsilon] \cup [1+\epsilon, 2-2\epsilon]$. Notice that $2\epsilon - 1 \leq x - y < 0$. Therefore,

$$|\sin(\pi x) - \sin(\pi y)|^2 = 4\cos^2\left(\frac{\pi(x+y)}{2}\right)\sin^2\left(\frac{\pi(x-y)}{2}\right) \geq 4\sin^2\left(\frac{\pi\epsilon}{2}\right)|x-y|^2,$$

because $\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1$ holds for all $\theta \in [-\frac{\pi}{2}, 0)$.

Case 2: $x + y \in [1-\epsilon, 1+\epsilon]$. In this case, we have $|\cos(\pi(x+y))| \geq \cos(\pi\epsilon)$ and $\epsilon - 1 \leq x - y < 0$. This implies that $|\sin(\pi(x-y))| \geq \frac{\sin(\pi(1-\epsilon))}{\pi(1-\epsilon)}\pi|x-y|$, and hence

$$\begin{aligned}|\sin(2\pi x) - \sin(2\pi y)|^2 &= 4\cos^2(\pi(x+y))\sin^2(\pi(x-y)) \\ &\geq 4\cos^2(\pi\epsilon)\frac{\sin^2(\pi(1-\epsilon))}{(1-\epsilon)^2}|x-y|^2.\end{aligned}$$

Combining *Case 1* and *Case 2*, the proof of (24) is completed.

We now turn to the proof of (25). Recall that $e_i^N, i = 1, 2$, defined in (11) is the linear interpolation of points $\{e_i(j/N),\ j \in \mathbb{Z}_{N+1} \cup \{0\}\}$. We split the interval $[\epsilon, 1-\epsilon]$ into

$$\left[\epsilon, \frac{1}{2} - \frac{1}{N}\right] \cup \left(\frac{1}{2} - \frac{1}{N}, \frac{1}{2} + \frac{1}{N}\right) \cup \left[\frac{1}{2} + \frac{1}{N}, 1-\epsilon\right] =: A_1 \cup A_2 \cup A_3.$$

For $N \geq 10$, $A_2 \subset [\frac{1}{4} + \frac{3}{2N}, \frac{3}{4} - \frac{3}{2N}]$. Without loss of generality, we also assume $N \geq \frac{2}{1-2\epsilon}$ (when $N < \frac{2}{1-2\epsilon}$, $A_3 = A_1 = \emptyset$, thus it suffices to consider the case of $x, y \in A_2$).

If $x, y \in [x_{l-1}, x_l]$ for some $l \in \mathbb{Z}_{N+1}$, then

$$|e_i^N(x) - e_i^N(y)| = N|x-y|\left|e_i\left(\frac{l-1}{N}\right) - e_i\left(\frac{l}{N}\right)\right|, \ i = 1, 2,$$

which along with (24) implies that

$$|e_1^N(x) - e_1^N(y)| + |e_2^N(x) - e_2^N(y)| \geq c(\epsilon)|x-y|.$$

Hence, it remains to prove the case of $x \in [x_l, x_{l+1})$ and $y \in (x_m, x_{m+1}]$ for some $l < m$, i.e., x_l is the largest grid point smaller than or equal to x and x_{m+1} is the smallest grid point greater than or equal to y.

(a) $x, y \in A_1$. Notice that $e_1(x) = \sqrt{2}\sin(\pi x)$ is monotone increasing on $[0, \frac{1}{2} - \frac{1}{2N}]$ with derivative $e_1'(x) = \sqrt{2}\pi\cos(\pi x) \in [\sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right), \sqrt{2}\pi]$. By the mean value theorem, we have for any $z_1, z_2 \in [0, \frac{1}{2} - \frac{1}{2N}]$ with $z_1 < z_2$

$$e_1(z_2) - e_1(z_1) \geq \sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right)(z_2 - z_1). \tag{26}$$

The monotone increasing property of e_1 on $[0, \frac{1}{2}]$ implies that e_1^N is also monotone increasing on $[0, \frac{1}{2} - \frac{1}{N}]$. Hence, for $0 \leq x_l \leq x < x_{l+1} \leq x_m < y \leq x_{m+1} \leq \frac{1}{2}$, it holds that $e_1^N(y) \geq e_1^N(x_m) \geq e_1^N(x_{l+1}) \geq e_1^N(x)$, and thus

$$\begin{aligned}
|e_1^N(x) - e_1^N(y)| &= e_1^N(y) - e_1^N(x) \\
&= \Big(e_1^N(x_{l+1}) - e_1^N(x)\Big) + \Big(e_1^N(x_m) - e_1^N(x_{l+1})\Big) + \Big(e_1^N(y) - e_1^N(x_m)\Big) \\
&= N(x_{l+1} - x)\Big(e_1(x_{l+1}) - e_1(x_l)\Big) + \Big(e_1(x_m) - e_1(x_{l+1})\Big) \\
&\quad + N(y - x_m)\Big(e_1(x_{m+1}) - e_1(x_m)\Big),
\end{aligned}$$

where in the last step, we have used (11), $x \in [x_l, x_{l+1})$, and $y \in (x_m, x_{m+1}]$. If N is odd, then the relations $y \leq \frac{1}{2} - \frac{1}{N} < \frac{1}{2} - \frac{1}{2N} = \frac{N-1}{2N}$ and $y \in (x_m, x_{m+1}]$ imply $x_{m+1} \leq \frac{N-1}{2N} = \frac{1}{2} - \frac{1}{2N}$. If N is even, then $y \leq \frac{1}{2} - \frac{1}{N} = \frac{\frac{N}{2}-1}{N}$ implies $x_{m+1} \leq \frac{1}{2} - \frac{1}{N}$. Thus, we have $0 \leq x_l \leq x < x_{l+1} \leq x_m < y \leq x_{m+1} \leq \frac{1}{2} - \frac{1}{2N}$, which together with (26) gives

$$\begin{aligned}
|e_1^N(x) - e_1^N(y)| &= N(x_{l+1} - x)\Big(e_1(x_{l+1}) - e_1(x_l)\Big) + \Big(e_1(x_m) - e_1(x_{l+1})\Big) \\
&\quad + N(y - x_m)\Big(e_1(x_{m+1}) - e_1(x_m)\Big) \\
&\geq (x_{l+1} - x)\sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right) + \sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right)(x_m - x_{l+1}) \\
&\quad + (y - x_m)\sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right) \\
&= \sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right)|y - x|.
\end{aligned}$$

(b) $x, y \in [\frac{1}{4} + \frac{3}{2N}, \frac{3}{4} - \frac{3}{2N}]$ or $x, y \in A_3$. Notice that $e_1(x) = \sqrt{2}\sin(\pi x)$ is monotone decreasing on $[\frac{1}{2} + \frac{1}{2N}, 1]$ with derivative $e_1'(x) = \sqrt{2}\pi\cos(\pi x) \in [-\sqrt{2}\pi, -\sqrt{2}\pi\sin\left(\frac{\pi}{2N}\right)]$, and $e_2(x) = \sqrt{2}\sin(2\pi x)$ is monotone decreasing on $[\frac{1}{4} + \frac{1}{2N}, \frac{3}{4} - \frac{1}{2N}]$ with derivative $e_2'(x) = 2\sqrt{2}\pi\cos(2\pi x) \in [-2\sqrt{2}\pi, -2\sqrt{2}\pi\sin\left(\frac{\pi}{N}\right)]$. Therefore,

$$|e_1(z_1) - e_1(z_2)| \geq \sqrt{2}\pi \sin\left(\frac{\pi}{2N}\right)|z_1 - z_2|, \quad \forall\, z_1, z_2 \in \left[\frac{1}{2} + \frac{1}{2N}, 1\right], \tag{27}$$

$$|e_2(z_1) - e_2(z_2)| \geq 2\sqrt{2}\pi \sin\left(\frac{\pi}{N}\right)|z_1 - z_2|, \quad \forall\, z_1, z_2 \in \left[\frac{1}{4} + \frac{1}{2N}, \frac{3}{4} - \frac{1}{2N}\right]. \tag{28}$$

If $x, y \in [\frac{1}{4} + \frac{3}{2N}, \frac{3}{4} - \frac{3}{2N}]$, then $\frac{1}{4} + \frac{1}{2N} \leq x_l \leq x < x_{l+1} \leq x_m < y \leq x_{m+1} \leq \frac{3}{4} - \frac{1}{2N}$. Thus, similar as in (a), the monotone decreasing property of e_2^N on $[\frac{1}{4} + \frac{3}{2N}, \frac{3}{4} - \frac{3}{2N}]$ and (28) produce

$$
\begin{aligned}
|e_2^N(x)-e_2^N(y)| &= |e_2^N(x)-e_2^N(x_{l+1})|+|e_2^N(x_{l+1})-e_2^N(x_m)|+|e_2^N(x_m)-e_2^N(y)|\\
&\geq 2\sqrt{2}\pi\sin\left(\frac{\pi}{N}\right)(x_{l+1}-x)+2\sqrt{2}\pi\sin\left(\frac{\pi}{N}\right)(x_m-x_{l+1})\\
&\quad +2\sqrt{2}\pi\sin\left(\frac{\pi}{N}\right)(y-x_m)\\
&=2\sqrt{2}\pi\sin\left(\frac{\pi}{N}\right)|x-y|.
\end{aligned}
$$

If $x,y\in A_3$, then it follows from $x_l\leq x<x_{l+1}$ that for odd N, $x_l\geq\frac{N+1}{2N}$ due to $x\geq\frac{1}{2}+\frac{1}{N}>\frac{1}{2}+\frac{1}{2N}=\frac{N+1}{2N}$, and for even N, $x_l\geq\frac{1}{2}+\frac{1}{N}$ due to $x\geq\frac{1}{2}+\frac{1}{N}=\frac{N+2}{2N}$. Thus, it holds that $\frac{1}{2}+\frac{1}{2N}\leq x_l\leq x<x_{l+1}\leq x_m<y\leq x_{m+1}\leq 1$, which along with (27) gives

$$
\begin{aligned}
|e_1^N(x)-e_1^N(y)| &= |e_1^N(x)-e_1^N(x_{l+1})|+|e_1^N(x_{l+1})-e_1^N(x_m)|+|e_1^N(x_m)-e_1^N(y)|\\
&\geq \sqrt{2}\pi\sin\left(\frac{\pi}{2N}\right)(x_{l+1}-x)+\sqrt{2}\pi\sin\left(\frac{\pi}{2N}\right)(x_m-x_{l+1})\\
&\quad +\sqrt{2}\pi\sin\left(\frac{\pi}{2N}\right)(y-x_m)\\
&=\sqrt{2}\pi\sin\left(\frac{\pi}{2N}\right)|x-y|.
\end{aligned}
$$

(c) $x\in A_1,\ y\in A_2$. If $x\in[\frac{1}{4}+\frac{3}{2N},\frac{3}{4}-\frac{3}{2N}]\cap A_1$, then $x,y\in[\frac{1}{4}+\frac{3}{2N},\frac{3}{4}-\frac{3}{2N}]$, and (25) holds by virtue of (b). If $x\in[\epsilon,\frac{1}{4}+\frac{3}{2N})$, then $e_1^N(x)\leq\sqrt{2}\sin\left((\frac{1}{4}+\frac{3}{2N})\pi\right)$. Besides, $y\in A_2:=\left(\frac{1}{2}-\frac{1}{N},\frac{1}{2}+\frac{1}{N}\right)$ implies that $e_1^N(y)\geq\sqrt{2}\sin\left(\pi\left(\frac{1}{2}-\frac{2}{N}\right)\right)>e_1^N(x)$ for $N>14$. Hence,

$$
\begin{aligned}
|e_1^N(x)-e_1^N(y)| &\geq \sqrt{2}\sin\left(\pi\left(\frac{1}{2}-\frac{2}{N}\right)\right)-\sqrt{2}\sin\left(\left(\frac{1}{4}+\frac{3}{2N}\right)\pi\right)\\
&=2\sqrt{2}\cos\left(\left(\frac{3}{8}-\frac{1}{4N}\right)\pi\right)\sin\left(\left(\frac{1}{8}-\frac{7}{4N}\right)\pi\right)\\
&\geq 2\sqrt{2}\cos\left(\frac{3\pi}{8}\right)\sin\left(\left(\frac{1}{8}-\frac{7}{4N}\right)\pi\right)|x-y|,
\end{aligned}
$$

since $|x-y|\leq 1$ and $N>14$.

(d) $x\in A_2,\ y\in A_3$. In this case, the proof is similar to (c).

(e) $x\in A_1,\ y\in A_3$. For $x\in A_1$, $e_2^N(x)\geq\sqrt{2}\min\left\{\sin(2\pi\epsilon),\sin(\frac{2\pi}{N})\right\}=:c_0>0$, and $e_2^N(y)\leq -c_0$. Hence,

$$
|e_2^N(x)-e_2^N(y)|\geq 2c_0\geq 2c_0|x-y|.
$$

The proof is finished. □

The condition $N>14$ in Lemma 3.2 is only considered for technical reasons and may be not necessary. Anyway, for a proper approximation, the partition parameter N is always required to be large enough. Based on Lemma 3.2, we proceed to obtain the optimal Hölder continuity exponent of $(t,x)\mapsto v^N(t,x)$ for SGM or FDM. In view of (10) and (12), we write the discrete heat kernel associated with FDM or SGM into the following unified formulation

$$
G_t^N(x,y)=\sum_{k=1}^{N-1}\exp(\lambda_k^N t)\varphi_k^N(x)\psi_k^N(y), \tag{29}
$$

where $\lambda_k^N=-k^2\pi^2$, $\varphi_k^N(x)=e_k(x)$, $\psi_k^N(y)=e_k(y)$ for the case of SGM, and $\lambda_k^N=-4N^2\sin^2\left(\frac{k}{2N}\pi\right)$, $\varphi_k^N(x)=e_k^N(x)$, $\psi_k^N(y)=e_k(\kappa_N(y))$ for the case of FDM.

Proposition 3.3. *Let* $0 < \epsilon < \frac{1}{2}$ *and* $N > 14$. *Then there exist positive constants* $C_i = C_i(N, \epsilon, T_0, T)$, $i = 1, 2$, *such that for any* (t, x), $(s, y) \in [T_0, T] \times [\epsilon, 1-\epsilon]$,

$$C_1(|t-s| + |x-y|^2) \leq \mathbb{E}|v^N(t,x) - v^N(s,y)|^2 \leq C_2(|t-s| + |x-y|^2). \tag{30}$$

Proof. The proof is separated into three steps.

Step 1: One can check that the discrete heat kernel (29) associated with SGM or FDM satisfies the following two facts:

(i) the sequence $\{\lambda_k^N\}_{k\in\mathbb{Z}_N} \subset (-\infty, 0]$ is strictly decreasing with respect to k;

(ii) for every $N \geq 1$ and $k \in \mathbb{Z}_N$, functions φ_k^N, $\psi_k^N : [0,1] \to \mathbb{R}$ are uniformly bounded from below and above by $-\sqrt{2}$ and $\sqrt{2}$, respectively. Moreover,

$$\left|\varphi_k^N(x) - \varphi_k^N(y)\right| \leq \sqrt{2}\pi k|x-y|, \quad \forall\, x, y \in [0,1].$$

Then the proof of the right side of (30) is standard by using the above facts (i) and (ii).

Step 2: In this step, we prove the left side of (30) for $t = s$ or $x = y$.

We first prove the left side of (30) for the case of $x = y$. Without loss of generality, assume that $t \geq s$. Notice that

$$\int_0^1 \psi_m^N(y)\psi_n^N(y)\mathrm{d}y = \delta_{m,n}, \tag{31}$$

which leads to

$$\begin{aligned}\mathbb{E}|v^N(t,x) - v^N(s,x)|^2 &= \int_0^s\int_0^1 |G_{t-r}^N(x,z) - G_{s-r}^N(x,z)|^2\mathrm{d}z\mathrm{d}r + \int_s^t\int_0^1 |G_{t-r}^N(x,z)|^2\mathrm{d}z\mathrm{d}r \\ &\geq \sum_{k=1}^{N-1}\int_s^t e^{2\lambda_k^N(t-r)}\mathrm{d}r|\varphi_k^N(x)|^2. \end{aligned}\tag{32}$$

Noticing that for any $x \in [\epsilon, 1-\epsilon]$, we have $\varphi_1^N(x) \geq \varphi_1^N(\epsilon) > 0$. This yields that (32) is bounded from below as

$$\mathbb{E}|v^N(t,x) - v^N(s,x)|^2 \geq \int_s^t e^{2\lambda_1^N(t-r)}\mathrm{d}r|\varphi_1^N(x)|^2 \geq C(\epsilon, T, N)(t-s), \tag{33}$$

which proves the lower bound in (30) for the case of $x = y$.

To prove the case of $t = s \geq T_0$, it is sufficient to notice that

$$\begin{aligned}\mathbb{E}|v^N(t,x) - v^N(t,y)|^2 &= \int_0^t\int_0^1 |G_{t-r}^N(x,z) - G_{t-r}^N(y,z)|^2\mathrm{d}z\mathrm{d}r \\ &= \sum_{k=1}^{N-1}\int_0^t e^{2\lambda_k^N(t-r)}\mathrm{d}r|\varphi_k^N(x) - \varphi_k^N(y)|^2,\end{aligned}$$

and

$$|\varphi_1^N(x) - \varphi_1^N(y)|^2 + |\varphi_2^N(x) - \varphi_2^N(y)|^2 \geq c(\epsilon, N)|x-y|^2,$$

by virtue of Lemma 3.2. It follows that

$$\begin{aligned}\mathbb{E}|v^N(t,x)-v^N(t,y)|^2 &\geq \sum_{k=1}^{2}\int_0^t e^{2\lambda_k^N(t-r)}\mathrm{d}r|\varphi_k^N(x)-\varphi_k^N(y)|^2\\ &\geq \frac{e^{2\lambda_2^N t}-1}{2\lambda_2^N}\sum_{k=1}^{2}|\varphi_k^N(x)-\varphi_k^N(y)|^2 \geq c(\epsilon,T_0,N)|x-y|^2.\end{aligned}$$

Step 3: In *Step 1* and *Step 2*, we have shown that there exist K_i, $i=1,2,3,4$ such that for any $t\in[T_0,T]$,

$$K_1|x-y|^2 \leq \mathbb{E}|v^N(t,x)-v^N(t,y)|^2 \leq K_2|x-y|^2, \quad \forall\, x,y\in[\epsilon,1-\epsilon],$$

and that for any $x\in[\epsilon,1-\epsilon]$,

$$K_3|t-s| \leq \mathbb{E}|v^N(t,x)-v^N(s,x)|^2 \leq K_4|t-s|, \quad \forall\, t,s\in[T_0,T].$$

In order to extend the lower bound in (30) to the case of $t\neq s$ and $x\neq y$, we consider the following two situations.

Case 1: If $|x-y|^2 \geq \frac{4K_4}{K_1}|t-s|$, then by the inequality $(a+b)^2\geq \frac{1}{2}a^2-b^2$ for $a,b\in\mathbb{R}$,

$$\begin{aligned}\mathbb{E}|v^N(t,x)-v^N(s,y)|^2 &\geq \frac{1}{2}\mathbb{E}|v^N(t,x)-v^N(t,y)|^2-\mathbb{E}|v^N(t,y)-v^N(s,y)|^2\\ &\geq \frac{K_1}{2}|x-y|^2-K_4|t-s| \geq \frac{K_1}{4}|x-y|^2\\ &\geq \frac{K_1}{8}\left(|x-y|^2+\frac{4K_4}{K_1}|t-s|\right)\\ &\geq \min\left\{\frac{K_1}{8},\frac{K_4}{2}\right\}(|x-y|^2+|t-s|).\end{aligned}$$

Case 2: If $\frac{K_1}{4K_4}|x-y|^2\leq|t-s|$ (assume without loss of generality $t>s$), then similar to (33), it holds for some $C=C(\epsilon,T)>0$ that

$$\begin{aligned}\mathbb{E}|v^N(t,x)-v^N(s,y)|^2 &\geq \int_s^t\int_0^1 |G_{t-r}^N(x,z)|^2\mathrm{d}z\mathrm{d}r \geq C|t-s|\\ &\geq \frac{C}{2}(|t-s|+\frac{K_1}{4K_4}|x-y|^2)\\ &\geq \min\left\{\frac{C}{2},\frac{CK_1}{8K_4}\right\}(|t-s|+|x-y|^2).\end{aligned}$$

The proof is finished. □

Proof of Theorem 2.3. We will apply Proposition 3.1 with $I=[T_0,T]$ to prove Theorem 2.3. By Proposition 3.3 with $x=y$, $v^N(\cdot,x)$ satisfies (C1) of Proposition 3.1 with $H=\frac{1}{2}$. For any fixed $x\in[\epsilon,1-\epsilon]$,

$$\operatorname{Var} v^N(t,x)=\sum_{k=1}^{N-1}\int_0^t e^{2\lambda_k^N(t-r)}\mathrm{d}r|\varphi_k^N(x)|^2, \quad \forall\, t>0.$$

Thus, $\operatorname{Var} v^N(\cdot,x)$ is a Lipschitz function with respect to $t\in[T_0,T]$, i.e., (18) holds with $\eta=1$ and $X_0(\cdot)=v^N(\cdot,x)$. In order to verify (19), it suffices to show that for $t,s\in[T_0,T]$ with $t\neq s$,

$$\mathbb{E}|v^N(t,x)|^2\,\mathbb{E}|v^N(s,x)|^2-\left|\mathbb{E}[v^N(t,x)v^N(s,x)]\right|^2>0, \quad \forall\, x\in[\epsilon,1-\epsilon]. \tag{34}$$

Indeed, assume without loss of generality $t > s$. Then (31) and the Hölder inequality imply

$$\begin{aligned}\left|\mathbb{E}[v^N(t,x)v^N(s,x)]\right|^2 &= \left(\int_0^s\int_0^1 G^N_{t-r}(x,z)G^N_{s-r}(x,z)\mathrm{d}z\mathrm{d}r\right)^2\\ &\le \left(\int_0^s\int_0^1 |G^N_{t-r}(x,z)|^2\mathrm{d}z\mathrm{d}r\right)\left(\int_0^s\int_0^1 |G^N_{s-r}(x,z)|^2\mathrm{d}z\mathrm{d}r\right)\\ &< \mathbb{E}|v^N(t,x)|^2\,\mathbb{E}|v^N(s,x)|^2,\end{aligned}$$

where in the last step, we have used $t > s$. This proves that $v^N(\cdot,x)$ satisfies (C2) of Proposition 3.1. Besides, we notice that for any $t \ge T_0/2$,

$$\mathrm{Var}\, v^N(t,x) \ge \int_0^t e^{2\lambda_1^N(t-r)}\mathrm{d}r|\varphi_1^N(x)|^2 \ge \frac{e^{\lambda_1^N T_0}-1}{2\lambda_1^N}\left|\varphi_1^N(\epsilon)\right|^2 > 0, \quad \forall\, x\in[\epsilon, 1-\epsilon], \tag{35}$$

which implies that $v^N(\cdot,x)$ satisfies (C0) with $\delta = T_0/2$. In conclusion, we have shown that $v^N(\cdot,x)$ satisfies (C0), (C1) and (C2) of Proposition 3.1, which completes the proof of Theorem 2.3. □

3.2. *Proof of Theorem 2.4*

In this part, let $v^N(t,x)$ be the numerical solution of SGM for (3). By (23), we have

$$\mathrm{Var}\left(v^N(t,x)|v^N(s,y)\right) = \frac{\mathrm{Var}\, v^N(t,x)\mathrm{Var}\, v^N(s,y) - \mathrm{Cov}(v^N(t,x), v^N(s,y))^2}{\mathrm{Var}\, v^N(s,y)}.$$

Based on (35), we proceed to derive the lower bound of $\mathrm{Var}\left(v^N(t,x)|v^N(s,y)\right)$.

Proposition 3.4. *Let $v^N(t,x)$ be the numerical solution of SGM for (3). Then for any $0<\epsilon<\frac{1}{2}$, there exists $N_0 := N_0(\epsilon)$ such that for all $N \ge N_0$,*

$$\mathrm{Var}\, v^N(t,x)\mathrm{Var}\, v^N(t,y) - \mathrm{Cov}(v^N(t,x), v^N(t,y))^2 \ge c(\epsilon, N, T_0)|x-y|^2$$

holds for any $x, y \in [\epsilon, 1-\epsilon]$ and $t\in[T_0, T]$.

Proof. Without loss of generality, assume that $\epsilon \le y < x \le 1-\epsilon$. Let $t \in [T_0, T]$ be arbitrarily fixed. First, we claim that for any $N \ge 4$,

$$\mathrm{Var}\, v^N(t,x)\mathrm{Var}\, v^N(t,y) - \mathrm{Cov}(v^N(t,x), v^N(t,y))^2 > 0, \quad \forall\, x \ne y. \tag{36}$$

Otherwise, there exist $x_0 \ne y_0$ (without loss of generality, assume $y_0 < x_0$) and $\lambda_0 \in \mathbb{R}$ such that $v^N(t,x_0) = \lambda_0 v^N(t,y_0)$ a.s. Hence,

$$\begin{aligned}&\int_0^t\int_0^1 |G^N_{t-r}(x_0,z) - \lambda_0 G^N_{t-r}(y_0,z)|^2\mathrm{d}z\mathrm{d}r\\ &= \sum_{k=1}^{N-1}\int_0^t e^{-2k^2\pi^2(t-r)}\mathrm{d}r|e_k(x_0)-\lambda_0 e_k(y_0)|^2 = 0,\end{aligned} \tag{37}$$

which implies that $\sin(k\pi x_0) = \lambda_0\sin(k\pi y_0)$ for all $1 \le k \le N-1$. Note that $\sin(\pi x) \ne 0$ for any $x \in [\epsilon, 1-\epsilon]$. Thus, $\lambda_0 = \frac{\sin(\pi x_0)}{\sin(\pi y_0)} = \frac{\sin(k\pi x_0)}{\sin(k\pi y_0)}$ provided $\sin(k\pi y_0) \ne 0$. We claim that under (37), $\sin(2\pi y_0) \ne 0$ and $\sin(3\pi y_0) \ne 0$.

In fact, if $\sin(2\pi y_0)=0$, then $\sin(2\pi x_0)=0$. This implies $x_0=y_0=\frac{1}{2}$, which contradicts with $x_0\neq y_0$. If $\sin(3\pi y_0)=0$, then $\sin(3\pi x_0)=0$. This implies $x_0, y_0\in\{\frac{1}{3},\frac{2}{3}\}$. Since $y_0<x_0$, it holds that $y_0=\frac{1}{3}$ and $x_0=\frac{2}{3}$. However, this contradicts with $\lambda_0=\frac{\sin(\pi x_0)}{\sin(\pi y_0)}=\frac{\sin(2\pi x_0)}{\sin(2\pi y_0)}$.

Since $\sin(2\pi y_0)\neq 0$ and $\sin(3\pi y_0)\neq 0$, we have

$$\lambda_0=\frac{\sin(\pi x_0)}{\sin(\pi y_0)}=\frac{\sin(2\pi x_0)}{\sin(2\pi y_0)}=\frac{\sin(3\pi x_0)}{\sin(3\pi y_0)}.$$

By the elementary identities $\sin(2\pi x_0)=2\sin(\pi x_0)\cos(\pi x_0)$ and $\sin(3\pi x_0)=3\sin(\pi x_0)-4\sin^3(\pi x_0)$, it must hold that $\cos(\pi x_0)=\cos(\pi y_0)$ and $\sin^2(\pi x_0)=\sin^2(\pi y_0)$. However, this only occurs when $x_0=y_0$ since $x_0, y_0\in(0,1)$, and hence (37) does not hold. Thus, we obtain (36).

By denoting

$$a_k=\left(\int_0^t e^{-2k^2\pi^2(t-r)}\mathrm{d}r\right)^{\frac{1}{2}}e_k(x),\ b_k=\left(\int_0^t e^{-2k^2\pi^2(t-r)}\mathrm{d}r\right)^{\frac{1}{2}}e_k(y),$$

we rewrite

$$\begin{aligned}&\mathrm{Var}\,v^N(t,x)\mathrm{Var}\,v^N(t,y)-\mathrm{Cov}(v^N(t,x),v^N(t,y))^2\\&=\left(\sum_{i=1}^{N-1}a_i^2\right)\left(\sum_{j=1}^{N-1}b_j^2\right)-\left|\sum_{i=1}^{N-1}a_ib_i\right|^2=\sum_{i<j}|a_ib_j-a_jb_i|^2.\end{aligned}\tag{38}$$

By definitions of a_k and b_k, we have

$$|a_ib_j-a_jb_i|^2=\frac{1}{4}K^N_{i,j}(t)|e_i(x)e_j(y)-e_j(x)e_i(y)|^2,$$

where

$$K^N_{i,j}(t):=\frac{(e^{-2\pi^2i^2t}-1)(e^{-2\pi^2j^2t}-1)}{\pi^4i^2j^2}.$$

Obviously, $0<K^N_{i,j}(T_0)\le K^N_{i,j}(t)\le K^N_{i,j}(T)<\infty,\ \forall t\in[T_0,T]$. Notice that

$$\begin{aligned}&\sin(i\pi x)\sin(j\pi y)-\sin(i\pi y)\sin(j\pi x)\\&=\sin\frac{(i+j)\pi(x-y)}{2}\sin\frac{(j-i)\pi(x+y)}{2}-\sin\frac{(i+j)\pi(x+y)}{2}\sin\frac{(j-i)\pi(x-y)}{2}.\end{aligned}\tag{39}$$

Choose $N_0:=N_0(\epsilon)>2$ such that $(2N_0-3)\sin(\pi\epsilon)\ge\frac{\pi}{2}+1$ and let $N\ge N_0$ be arbitrarily fixed. For any $y\in[\epsilon,1-\epsilon]$, denote $y^\epsilon_N:=y+\frac{1}{2N-3}$. By (38), we have

$$\begin{aligned}&\mathrm{Var}\,v^N(t,x)\mathrm{Var}\,v^N(t,y)-\mathrm{Cov}(v^N(t,x),v^N(t,y))^2\\&\ge|a_{N-2}b_{N-1}-a_{N-1}b_{N-2}|^2\\&\ge K^N_{N-2,N-1}(T_0)\left|\sin\frac{(2N-3)\pi(x-y)}{2}\sin\frac{\pi(x+y)}{2}-\sin\frac{(2N-3)\pi(x+y)}{2}\sin\frac{\pi(x-y)}{2}\right|^2.\end{aligned}$$

We are going to derive the lower bound of $\mathrm{Var}\,v^N(t,x)\mathrm{Var}\,v^N(t,y)-\mathrm{Cov}(v^N(t,x),v^N(t,y))^2$, which is separated into two cases.

Case 1: $y\in[\epsilon,1-\epsilon)$ and $x\in(y,y^\epsilon_N]\cap(\epsilon,1-\epsilon]$. We introduce

$$f_N(x,y):=\sin\frac{(2N-3)\pi(x-y)}{2}\sin\frac{\pi(x+y)}{2}-\sin\frac{(2N-3)\pi(x+y)}{2}\sin\frac{\pi(x-y)}{2}$$

$$\geq \sin\frac{(2N-3)\pi(x-y)}{2}\sin(\pi\epsilon)-\sin\frac{\pi(x-y)}{2}$$
$$\geq (2N-3)(x-y)\sin(\pi\epsilon)-\frac{\pi}{2}(x-y),$$

where we have used $0<x-y\leq\frac{1}{2N-3}$, and $\frac{2}{\pi}z\leq\sin(z)\leq z,\ \forall z\in(0,\frac{\pi}{2}]$. Hence, for any $N\geq N_0$ with $(2N_0-3)\sin(\pi\epsilon)-\frac{\pi}{2}\geq 1$,

$$f_N(x,y)\geq (x-y),$$

which implies that

$$\operatorname{Var} v^N(t,x)\operatorname{Var} v^N(t,y)-\operatorname{Cov}(v^N(t,x),v^N(t,y))^2\geq c(T_0,N,\epsilon)|x-y|^2. \tag{40}$$

Case 2: $y\in[\epsilon,1-\epsilon)$ and $x\in(y_N^{\epsilon},1-\epsilon]$. By (36) and the continuity of $\operatorname{Var} v^N(t,x)$ and $\operatorname{Cov}(v^N(t,x),v^N(t,y))$, we have that there is $c=c(\epsilon,N)$ such that

$$\operatorname{Var} v^N(t,x)\operatorname{Var} v^N(t,y)-\operatorname{Cov}(v^N(t,x),v^N(t,y))^2\geq c\geq\frac{c}{(1-2\epsilon)^2}|x-y|^2. \tag{41}$$

Combining (40) and (41), we finish the proof. □

Proof of Theorem 2.4. The proof of (16) is similar to that of Theorem 2.3. To prove (15), we set $I=[\epsilon,1-\epsilon]$. Similar as in (35), for any $x\in I^{\delta}$ with $\delta=\frac{\epsilon}{2}$, we have

$$\operatorname{Var} v^N(t,x)\geq\int_0^t e^{-2\pi^2(t-r)}\mathrm{d}r|\sqrt{2}\sin(\pi x)|^2\geq\frac{1-e^{-2\pi^2T_0}}{\pi^2}\sin^2\left(\pi\frac{\epsilon}{2}\right)>0,$$

where $t\in[T_0,T]$. This implies that $v^N(t,\cdot)$ satisfies (C0) with $\delta=\frac{\epsilon}{2}$. Propositions 3.3 and 3.4 show that $v^N(t,\cdot)$ satisfies (C1) and (C2)$'$ of Proposition 3.1, which completes the proof of (15). □

3.3. *Proof of Theorem 2.5*

Recall that the numerical solution v_M of EEM for (3) is defined in (13). We begin with giving the optimal Hölder continuity of $v_M(t_i,\cdot)$.

Lemma 3.5. *Let $0<\epsilon<\frac{1}{2}$ be fixed and $M\geq 3$. Then there exist positive constants $C_j=C_j(\epsilon,M,T_0,T),\ j=1,2$, such that for any $t_i\in[T_0,T]$,*

$$C_1|x-y|^2\leq\mathbb{E}|v_M(t_i,x)-v_M(t_i,y)|^2\leq C_2|x-y|^2,\quad\forall x,y\in[\epsilon,1-\epsilon].$$

Proof. First, it follows from (4), the orthogonality of $\{e_j\}_{j=1}^{\infty}$, and the mean value theorem that

$$\begin{aligned}\mathbb{E}|v_M(t_i,x)-v_M(t_i,y)|^2&=\int_0^{t_i}\int_0^1|G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z)-G_{t_i-[\frac{r}{\delta t}]\delta t}(y,z)|^2\mathrm{d}z\mathrm{d}r\\&=\sum_{j=0}^{i-1}\delta t\sum_{k=1}^{\infty}e^{-2k^2\pi^2(t_i-t_j)}|e_k(x)-e_k(y)|^2\\&\leq\sum_{j=1}^{i}\delta t\sum_{k=1}^{\infty}e^{-2k^2\pi^2t_j}2k^2\pi^2|x-y|^2.\end{aligned}$$

Since the function $x \mapsto xe^{-x}$ is monotone increasing on $[0,1]$ and monotone decreasing on $[1,\infty)$, and $f(1)=e^{-1}$, we have

$$\sum_{k=1}^{\infty} e^{-2k^2\pi^2\delta t}2k^2\pi^2\delta t \le \sum_{k\le \frac{1}{\sqrt{2\pi^2\delta t}}-1} e^{-2k^2\pi^2\delta t}2k^2\pi^2\delta t + e^{-1} + \sum_{k> \frac{1}{\sqrt{2\pi^2\delta t}}} e^{-2k^2\pi^2\delta t}2k^2\pi^2\delta t$$

$$\le \int_0^{\frac{1}{\sqrt{2\pi^2\delta t}}} e^{-2x^2\pi^2\delta t}2x^2\pi^2\delta t \mathrm{d}x + \int_{\frac{1}{\sqrt{2\pi^2\delta t}}}^{\infty} e^{-2x^2\pi^2\delta t}2x^2\pi^2\delta t \mathrm{d}x + e^{-1}.$$

By the change of variables $z = x\pi\sqrt{\delta t}$,

$$\int_0^{\infty} e^{-2x^2\pi^2\delta t}2x^2\pi^2\delta t \mathrm{d}x = \frac{1}{\pi\sqrt{\delta t}}\int_0^{\infty} e^{-2z^2}2z^2\mathrm{d}z \le \frac{C}{\sqrt{\delta t}},$$

where $C := \frac{1}{\pi}\int_0^{\infty} e^{-2z^2}2z^2\mathrm{d}z < \infty$. Therefore, for any $t_i \in [0,T]$

$$\mathbb{E}|v_M(t_i,x) - v_M(t_i,y)|^2 \le \sum_{j=1}^{i}\delta t\sum_{k=1}^{\infty} e^{-2k^2\pi^2\delta t}2k^2\pi^2|x-y|^2 \le \frac{T}{\delta t}\left(\frac{C}{\sqrt{\delta t}} + e^{-1}\right)|x-y|^2.$$

On the other hand, by the spectral expansion of G and Lemma 3.2,

$$\begin{aligned}\mathbb{E}|v_M(t_i,x) - v_M(t_i,y)|^2 &= \int_0^{t_i}\int_0^1 \left|G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z) - G_{t_i-[\frac{r}{\delta t}]\delta t}(y,z)\right|^2 \mathrm{d}z\mathrm{d}r \\ &\ge \sum_{k=1}^{2}\int_0^{t_i} e^{-2k^2\pi^2(t_i-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r|e_k(x)-e_k(y)|^2 \\ &\ge c(\epsilon, T_0, T)|x-y|^2.\end{aligned}$$

The proof is completed. $\square$

Proof of Theorem 2.5. We will apply Proposition 3.1 with $I = [\epsilon, 1-\epsilon]$ and $\delta = \frac{\epsilon}{2}$ to prove Theorem 2.5. For any $x \in I^{\delta} = [\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}]$, $t_i \in [T_0, T]$,

$$\operatorname{Var} v_M(t_i,x) = \sum_{k=1}^{\infty}\int_0^{t_i} e^{-2\pi^2k^2(t_i-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r|e_k(x)|^2 \ge c\sin^2\left(\frac{\pi\epsilon}{2}\right), \tag{42}$$

with $c = 2\int_0^{T_0} e^{-2\pi^2 T}\mathrm{d}r > 0$. This means that $v_M(t,\cdot)$ satisfies (C0) of Proposition 3.1 with $\delta = \frac{\epsilon}{2}$.

By introducing

$$a_k = \left(\int_0^t e^{-2k^2\pi^2(t-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r\right)^{\frac{1}{2}} e_k(x),\ b_k = \left(\int_0^t e^{-2k^2\pi^2(t-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r\right)^{\frac{1}{2}} e_k(y),\ k \in \mathbb{N}_+$$

and

$$K^N_{i,j}(t) := \left(\int_0^t e^{-2i^2\pi^2(t-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r\right)\left(\int_0^t e^{-2j^2\pi^2(t-[\frac{r}{\delta t}]\delta t)}\mathrm{d}r\right),\ i,j \in \mathbb{N}_+,$$

and repeating the proof of Proposition 3.4, one can verify that

$$\mathrm{Var}\, v_M(t_i,x)\mathrm{Var}\, v_M(t_i,y)-\mathrm{Cov}(v_M(t_i,x),v_M(t_i,y))^2\ge c(T_0,\epsilon)|x-y|^2,\quad \forall x,y\in[\epsilon,1-\epsilon],$$

which implies the condition (C2)′ of Proposition 3.1. In addition, Lemma 3.5 shows that $v_M(t,\cdot)$ satisfies (C1) of Proposition 3.1. The proof is completed by applying Proposition 3.1. □

4. Discussion on continuous versions of time discretizations

In this section, we consider the comparison of the continuous version of time discretization for the system of linear stochastic parabolic equations (2) and that of the system of linear stochastic differential equations. It turns out that the continuous EEM numerical solution $U_M(t,x)$ is smoother in every subinterval (t_i,t_{i+1}) than in grid points, which leads to the nonexistence of the lower bound of the Hölder exponent, while the continuous EEM numerical solution of the system of finite dimensional Ornstein–Uhlenbeck equations preserves the optimal Hölder exponent and the critical dimension of the exact solution.

4.1. Continuous version of time discretization in infinite dimensional case

Interpolation is usually used to extend the numerical solution from grid points to the whole interval. By (13), it is natural to define the continuous EEM numerical solution by

$$v_M(t,x)=\int_0^t\int_0^1 G_{t-[\frac{r}{\delta t}]\delta t}(x,z)W(\mathrm{d}r,\mathrm{d}z). \tag{43}$$

In the same way, we obtain the continuous EEM numerical solution $U_M(t,x)$ of system (2). We first study the Hölder continuity of $v_M(t,x)$ in time direction, which is crucial to the analysis of hitting probabilities of $U_M(\cdot,x)$.

Lemma 4.1. *Let v_M given by (13) be the numerical solution of EEM for (3). Then there exist positive constants $C_i=C_i(T_0,T,\epsilon)$, $i=1,2$, such that for any $0<t_j<t_i\le T$,*

$$C_1\sqrt{t_i-t_j}\le \mathbb{E}|v_M(t_i,x)-v_M(t_j,x)|^2\le C_2\sqrt{t_i-t_j}, \tag{44}$$

where $x\in[\epsilon,1-\epsilon]$.

Proof. In view of (13) and $t_j<t_i$, we have

$$\begin{aligned}&\mathbb{E}|v_M(t_i,x)-v_M(t_j,x)|^2\\ &=\int_0^{t_j}\int_0^1 |G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z)-G_{t_j-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r+\int_{t_j}^{t_i}\int_0^1 |G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r.\end{aligned} \tag{45}$$

Thus, to prove the lower bound of (44), it suffices to prove that there is $C_1>0$ such that

$$\sum_{k=j}^{i-1}\int_{t_k}^{t_{k+1}}\int_0^1 \left|G_{t_i-t_k}(x,z)\right|^2\mathrm{d}z\mathrm{d}r\ge C_1\sqrt{t_i-t_j}. \tag{46}$$

In fact, by the elementary property $\int_0^1|G_r(x,z)|^2\,\mathrm{d}z=G_{2r}(x,x)$ and Lemma 2.1,

$$
\begin{aligned}
&\sum_{k=j}^{i-1}\int_{t_k}^{t_{k+1}}\int_0^1 \left|G_{t_i-t_k}(x,z)\right|^2 \mathrm{d}z\mathrm{d}r = \sum_{k=j}^{i-1}\int_{t_k}^{t_{k+1}} G_{2(t_i-t_k)}(x,x)\mathrm{d}r \\
&\ge C\sum_{k=j}^{i-1}\frac{\delta t}{\sqrt{t_i-t_k}} \ge C\sum_{k=1}^{i-j}\frac{\delta t}{\sqrt{k\delta t}} \ge \int_{t_1}^{t_{i-j+1}}\frac{C}{\sqrt{r}}\mathrm{d}r \\
&= C\left(\sqrt{t_{i-j+1}}-\sqrt{t_1}\right).
\end{aligned}
$$

For $j=i-1$, $\sqrt{t_{i-j+1}}-\sqrt{t_1}=(\sqrt{2}-1)\sqrt{t_i-t_j}$, and for $1\le j<i-1$, $\sqrt{t_{i-j+1}}-\sqrt{t_1}\ge\frac{1}{2}\sqrt{t_i-t_j}$. Thus, we obtain (46), which yields the left side of (44). Similarly, using (5) gives

$$
\begin{aligned}
\sum_{k=j}^{i-1}\int_{t_k}^{t_{k+1}}\int_0^1 \left|G_{t_i-t_k}(x,z)\right|^2 \mathrm{d}z\mathrm{d}r &\le \sum_{k=j}^{i-1}\int_{t_k}^{t_{k+1}} P_{2(t_i-t_k)}(x,x)\mathrm{d}r \\
&\le C\int_0^{t_{i-j}}\frac{1}{\sqrt{r}}\mathrm{d}r = 2C\sqrt{t_i-t_j}.
\end{aligned}
$$

For the right side of (44), it remains to estimate the first term on the right side of (45). By (4),

$$
\begin{aligned}
&\int_0^{t_j}\int_0^1 |G_{t_i-[\frac{r}{\delta t}]\delta t}(x,z)-G_{t_j-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r \\
&=\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}}\sum_{k=1}^{\infty}|e^{-k^2\pi^2(t_i-t_m)}-e^{-k^2\pi^2(t_j-t_m)}|^2\mathrm{d}r e_k(x)^2 \\
&\le\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}}\sum_{k=1}^{\infty}|e^{-k^2\pi^2(t_i-r)}-e^{-k^2\pi^2(t_j-r)}|^2\mathrm{d}r e_k(x)^2 \\
&=\int_0^{t_j}\int_0^1 |G_{t_i-r}(x,z)-G_{t_j-r}(x,z)|^2\mathrm{d}z\mathrm{d}r \\
&\le C|t_i-t_j|^{\frac{1}{2}},
\end{aligned}
$$

where in the last step, we have applied [21, Lemma A1.1 (b)]. The proof is completed. □

The following corollary indicates that $\frac{1}{4}$ is the Hölder exponent of $v_M(\cdot,x)$ but is not the optimal Hölder exponent.

Lemma 4.2. *Let the condition of Lemma 4.1 hold and fix $x\in[\epsilon,1-\epsilon]$. Then there exists some positive constant $c_3=c_3(T_0,T)$ such that for any $T_0\le s<t\le T$,*

$$
\mathbb{E}|v_M(t,x)-v_M(s,x)|^2\le c_3\sqrt{t-s}. \tag{47}
$$

However, for any $H\in(0,\frac{1}{2})$, there is no $c_4>0$ such that for any $T_0\le s<t\le T$,

$$
\mathbb{E}|v_M(t,x)-v_M(s,x)|^2\ge c_4|t-s|^{2H}. \tag{48}
$$

Proof. Notice that

$$\begin{aligned}
&\mathbb{E}|v_M(t,x)-v_M(s,x)|^2\\
&=\int_0^s\int_0^1 |G_{t-[\frac{r}{\delta t}]\delta t}(x,z)-G_{s-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r+\int_s^t\int_0^1 |G_{t-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r\\
&=: A(s,t)+B(s,t).
\end{aligned}$$

Since $0<s<t\le T$, there is $j\le i+1$ such that $t\in[t_i,t_{i+1})$ and $s\in[t_{j-1},t_j)$.

If $i=j-1$, then $t_i\le s<t\le t_{i+1}$. Similar to the first term on the right side of (45), we also have $A(s,t)\le C\sqrt{t-s}$, which together with the fact $B(s,t)\le C\frac{t-s}{\sqrt{t-t_i}}\le C\sqrt{t-s}$ completes the proof of (47).

If $i\ge j$, then by Lemma 4.1,

$$\begin{aligned}
&\mathbb{E}|v_M(t,x)-v_M(s,x)|^2\\
&\le 3\mathbb{E}|v_M(t,x)-v_M(t_i,x)|^2+3\mathbb{E}|v_M(t_i,x)-v_M(t_j,x)|^2+3\mathbb{E}|v_M(t_j,x)-v_M(s,x)|^2\\
&\le C\sqrt{t-t_i}+c_1\sqrt{t_i-t_j}+C\sqrt{t_j-s}\le(2C+c_1)\sqrt{t-s}.
\end{aligned}$$

Assume by contradiction that there is $c_4>0$ such that (48) holds. Fix $t\in[t_i+\frac{\delta t}{2},t_{i+1})$ and let $s_n=t-\frac{1}{n}$ with positive integers $n>\frac{4}{\delta t}$, which ensures that $t>s_n\ge t_i+\frac{\delta t}{2}-\frac{1}{n}>t_i+\frac{\delta t}{4}$. By virtue of (4) and the uniform boundedness of $\{e_k\}_{k\ge1}$, it holds that

$$\begin{aligned}
A(s_n,t)&=\int_0^{s_n}\int_0^1 |G_{t-[\frac{r}{\delta t}]\delta t}(x,z)-G_{s_n-[\frac{r}{\delta t}]\delta t}(x,z)|^2\mathrm{d}z\mathrm{d}r\\
&\le 2\int_0^{s_n}\sum_{k=1}^{\infty}|e^{-k^2\pi^2(t-[\frac{r}{\delta t}]\delta t)}-e^{-k^2\pi^2(s_n-[\frac{r}{\delta t}]\delta t)}|^2\mathrm{d}r\\
&\le 2\int_0^{s_n}\sum_{k=1}^{\infty}e^{-2k^2\pi^2(s_n-[\frac{r}{\delta t}]\delta t)}|e^{-k^2\pi^2(t-s_n)}-1|^2\mathrm{d}r.
\end{aligned}$$

Since for $r\in[0,s_n]$, $s_n-[\frac{r}{\delta t}]\delta t\ge s_n-t_i>\frac{\delta t}{4}$, and $1-e^{-x}\le x$ for all $x>0$, we have

$$A(s_n,t)\le 2\int_0^{s_n}\sum_{k=1}^{\infty}e^{-k^2\pi^2\frac{\delta t}{2}}k^4\pi^4(t-s_n)^2\mathrm{d}r\le C(T,\delta t)|t-s_n|^2,$$

which indicates that $\lim_{n\to\infty}\frac{A(s_n,t)}{|t-s_n|^{2H}}=0$ for any $0<H<1$.

On the other hand, by (5) and the semigroup property of G,

$$B(s_n,t)=\int_{s_n}^t\int_0^1\left|G_{t-t_i}(x,z)\right|^2\mathrm{d}z\mathrm{d}r\le(t-s_n)P_{2(t-t_i)}(x,x)=\frac{t-s_n}{\sqrt{8\pi(t-t_i)}}.$$

Taking into account $t-t_i\ge\frac{\delta t}{2}$, we obtain that $\lim_{n\to\infty}\frac{B(s_n,t)}{|t-s_n|^{2H}}=0$ for any $0<H<\frac{1}{2}$. In conclusion, we have $\lim\limits_{n\to\infty}\frac{\mathbb{E}|v_M(t,x)-v_M(s_n,x)|^2}{|t-s_n|^{2H}}=0$, for $H\in(0,\frac{1}{2})$, which contradicts (48). The proof is finished. $\square$

By (47), we obtain the upper bound for hitting probabilities of $U_M(\cdot, x)$, i.e., for any $x \in [\epsilon, 1-\epsilon]$,

$$\mathbb{P}\{U_M([T_0, T] \times \{x\}) \cap A \neq \emptyset\} \leq C\mathscr{H}_{d-4}(A).$$

However, the nonexistence of the lower bound of the Hölder exponent in Lemma 4.2 prevents us from deriving the lower bound of hitting probabilities in time direction of $U_M(t, x)$ in terms of Bessel–Riesz capacity. This implies that the regularity of trajectories of $U_M(\cdot, x)$ is different from that of the exact solution $u(\cdot, x)$. However, for a system of finite dimensional Ornstein–Uhlenbeck equations, the result is different.

4.2. Continuous version of time discretization in finite dimensional case

Let $\{B(t) = (B^0(t), B^1(t), \ldots, B^d(t)), t \geq 0\}$ be a standard $(d+1)$-dimensional Brownian motion on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and $Y(t) = (Y^1(t), \ldots, Y^d(t))$ be the solution of the following system

$$\mathrm{d}Y^i(t) = -\lambda Y^i(t)\mathrm{d}t + \mathrm{d}B^i(t),\ t > 0,\ i = 1, \ldots, d, \tag{49}$$

where $\lambda > 0$ and $Y^i(0) = 0$. Obviously, each component $Y^i(t)$ is an independent copy of a 1-dimensional Ornstein–Uhlenbeck process $\{Y^0(t), t \geq 0\}$ which satisfies

$$\mathrm{d}Y^0(t) = -\lambda Y^0(t)\mathrm{d}t + \mathrm{d}B^0(t),\ t > 0, \tag{50}$$

and $Y^0(0) = 0$. Making use of Proposition 3.1, it is easy to verify that for any bounded Borel set A in $\mathbb{R}^d$,

$$C_1 \operatorname{Cap}_{d-2}(A) \leq \mathbb{P}\{Y([T_0, T]) \cap A \neq \emptyset\} \leq C_2\mathscr{H}_{d-2}(A)$$

with C_1, C_2 being positive constants depending on T_0, T, d, λ.

When we apply EEM to discretize (50) and use the same continuous approach as in (43), the associated numerical solution is

$$\bar{Y}^0(t) = \int_0^t e^{-\lambda(t-[\frac{r}{\delta t}]\delta t)}\mathrm{d}B^0(r),\ t \geq 0.$$

Analogous estimates yield that

$$C_1 \operatorname{Cap}_{d-2}(A) \leq \mathbb{P}\left\{\bar{Y}([T_0, T]) \cap A \neq \emptyset\right\} \leq C_2\mathscr{H}_{d-2}(A)$$

for any bounded Borel set A in $\mathbb{R}^d$, where $\bar{Y}$ is the continuous exponential Euler approximation of Y. It can be concluded that the continuous EEM numerical solution $\bar{Y} = \{\bar{Y}(t), t \in [T_0, T]\}$ for system (49) preserves the critical dimension of the exact solution $Y = \{Y(t), t \in [T_0, T]\}$, which is different from the infinite dimensional case. In fact, this property not only holds for the continuous EEM numerical solution, but also holds for the Euler–Maruyama method under a proper continuity approach. The Euler–Maruyama method applied to (50) yields

$$Y_i^0 = Y_{i-1}^0 - \lambda\delta t Y_{i-1}^0 + \triangle B_{i-1}^0,\ i \in \mathbb{N}_+,$$

where $\triangle B_i^0 = B^0(t_{i+1}) - B^0(t_i)$. After rearranging, we have

$$Y_i^0 = \sum_{k=0}^{i-1}(1-\lambda\delta t)^{i-1-k}\triangle B_k^0 = \int_0^{t_i}(1-\lambda\delta t)^{[\frac{t_i-r}{\delta t}]}\mathrm{d}B^0(r),\ i \in \mathbb{N}_+.$$

Naturally, we define the continuous Euler–Maruyama numerical solution for (50) by

$$\widetilde{Y}^0(t) = \int_0^t (1-\lambda\delta t)^{[\frac{t-r}{\delta t}]}\mathrm{d}B^0(r), \quad t \geq 0. \tag{51}$$

Proposition 4.3. *Fix $L>0$, $\delta t\in(0,\frac{1}{\lambda})$. Let $\widetilde{Y}(t)=(\widetilde{Y}^1(t),\ldots,\widetilde{Y}^d(t))$ be the continuous Euler–Maruyama numerical solution for system (49). Then there exist positive constants $C_i=C_i(T_0,T,d,\lambda)$, $i=1,2$, such that for any Borel set $A\subset[-L,L]^d$,*

$$C_1\,\mathrm{Cap}_{d-2}(A)\le\mathbb{P}\left\{\widetilde{Y}([T_0,T])\cap A\neq\emptyset\right\}\le C_2\mathscr{H}_{d-2}(A).$$

The proof of Proposition 4.3 can be found in Appendix A.3. In general, the hitting probabilities of continuous versions of numerical solutions depend on continuity approaches. If we consider the linear interpolation of the Euler–Maruyama numerical solution $\{Y_i^0,\ i\in\mathbb{N}_+\}$,

$$\widetilde{Y}^0(t)=\frac{t_i-t}{\delta t}Y_{i-1}^0+\frac{t-t_{i-1}}{\delta t}Y_i^0,\ t\in(t_{i-1},t_i),\ i\in\mathbb{N}_+,$$

then it can be verified that for some $C>0$,

$$\mathbb{E}|\widetilde{Y}^0(t)-\widetilde{Y}^0(s)|^2\le C(t-s),\quad\forall\,T\ge t>s\ge T_0.$$

However, we cannot obtain the existence of $c>0$ such that

$$\mathbb{E}|\widetilde{Y}^0(t)-\widetilde{Y}^0(s)|^2\ge c(t-s),\quad\forall\,T\ge t>s\ge T_0. \tag{52}$$

Actually, for any $t_m<s<t<t_{m+1}$,

$$\mathbb{E}|\widetilde{Y}^0(t)-\widetilde{Y}^0(s)|^2=\frac{(t-s)^2}{(\delta t)^2}\mathbb{E}|\widetilde{Y}^0(t_{i+1})-\widetilde{Y}^0(t_i)|^2\le C(\delta t,T,\lambda)(t-s)^2,$$

which contradicts with the lower bound in (52). Therefore, the linear interpolation may not be a proper choice to inherit the critical dimension of the exact solution.

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Appendix A

A.1. Proof of Lemma 2.1

Using (6), we have $G_t(x,x)\ge(1-2e^{-\frac{\epsilon^2}{t}})P_t(x,x)=\frac{1}{\sqrt{4\pi t}}(1-2e^{-\frac{\epsilon^2}{t}})$, which gives that

$$G_t(x,x)\ge\frac{1}{\sqrt{4\pi t}}(1-2e^{-\frac{\epsilon^2}{t}})\ge\frac{1}{2\sqrt{4\pi t}},\quad\forall\,x\in[\epsilon,1-\epsilon],\ t\in(0,c_\epsilon],$$

where $c_\epsilon:=\min\{\frac{\epsilon^2}{\log 4},T\}$. It follows from (4) that $G_t(x,x)\ge 2e^{-\pi^2T}\sin^2(\pi\epsilon)=:c$, $\forall\ (t,x)\in[c_\epsilon,T]\times[\epsilon,1-\epsilon]$, which indicates

$$G_t(x,x)\ge c\ge\frac{c}{\sqrt{t}}\sqrt{c_\epsilon},\quad\forall\,(t,x)\in[c_\epsilon,T]\times[\epsilon,1-\epsilon].$$

The proof is completed by choosing $C=\min\{c\sqrt{c_\epsilon},\frac{1}{2\sqrt{4\pi}}\}$.

A.2. Proof of Proposition 3.1

The proof of Proposition 3.1 is divided into two steps.

Step 1: We prove that (C0), (C1) and (C2) implies (20), which is mainly based on [1, Theorem 2.1]. In view of [1, Theorem 2.1], it suffices to prove that (C2) implies the following lower bound:

$$\mathrm{Var}\,(X_0(x)|X_0(y)) \geq c_4 \sum_{j=1}^{m} \left|x_j - y_j\right|^{2H_j}, \quad \forall\, x, y \in I. \tag{A.1}$$

From (21), (C0) and (22), we obtain that for any $x, y \in I$

$$\begin{aligned}&\mathrm{Var}(X_0(x)|X_0(y))\\ &\geq \frac{\left(\mathbb{E}|X_0(x)-X_0(y)|^2 - \left(\sqrt{\mathbb{E}|X_0(x)|^2} - \sqrt{\mathbb{E}|X_0(y)|^2}\right)^2\right)\left(4c_1 - \mathbb{E}|X_0(x)-X_0(y)|^2\right)}{4c_7}.\end{aligned} \tag{A.2}$$

We introduce the set

$$I_1 := \left\{(x, y) \in I \times I : \sum_{j=1}^{m} \left|x_j - y_j\right|^{2H_j} < \frac{3c_1}{c_3}, \quad \text{and} \quad \max_{1\leq j\leq m} |x_j - y_j|^{2H_j\eta} < \frac{2c_1c_2}{c_4^2 m}\right\}$$

and $I_2 := I \times I - I_1$, and divide the proof of (A.1) into two cases.

Case 1: $(x, y) \in I_1$.

By the upper bound of (C1), we obtain that for $(x, y) \in I_1$,

$$4c_1 - \mathbb{E}|X_0(x) - X_0(y)|^2 \geq 4c_1 - c_3 \sum_{j=1}^{m} \left|x_j - y_j\right|^{2H_j} \geq c_1. \tag{A.3}$$

Making use of (18) leads to

$$\left|\sqrt{\mathbb{E}|X_0(x)|^2} - \sqrt{\mathbb{E}|X_0(y)|^2}\right| = \frac{|\mathbb{E}|X_0(x)|^2 - \mathbb{E}|X_0(y)|^2|}{\sqrt{\mathbb{E}|X_0(x)|^2} + \sqrt{\mathbb{E}|X_0(y)|^2}} \leq \frac{c_4}{2\sqrt{c_1}} \sum_{j=1}^{m} \left|x_j - y_j\right|^{H_j(1+\eta)}. \tag{A.4}$$

Substituting (A.3) and (A.4) into (A.2) gives

$$\begin{aligned}\mathrm{Var}(X_0(x)|X_0(y)) &\geq \frac{c_1\left(\mathbb{E}|X_0(x)-X_0(y)|^2 - \frac{c_4^2}{4c_1}\left|\sum_{j=1}^m \left|x_j - y_j\right|^{H_j(1+\eta)}\right|^2\right)}{4c_7}\\ &\geq \frac{c_1\left(c_2\sum_{j=1}^m \left|x_j - y_j\right|^{2H_j} - \frac{c_4^2 m}{4c_1}\sum_{j=1}^m \left|x_j - y_j\right|^{2H_j(1+\eta)}\right)}{4c_7},\end{aligned}$$

thanks to the lower bound of (C1) and the Hölder inequality. Then an elementary calculation by using $\max_{1\leq j\leq m} |x_j - y_j|^{2H_j\eta} \leq \frac{2c_2c_1}{c_4^2 m}$ implies

$$\mathrm{Var}(X_0(x)|X_0(y)) \geq \frac{c_1c_2}{8c_7} \sum_{j=1}^{m} \left|x_j - y_j\right|^{2H_j}.$$

Case 2: $(x, y) \in I_2$.

In this case, it holds that $\sum_{j=1}^m \left|x_j - y_j\right|^{2H_j} \geq \min\left\{\frac{3c_1}{c_3}, (\frac{2c_1c_2}{c_4^2 m})^{\frac{1}{\eta}}\right\} =: c_9$. Making use of the identity (23) and the Hölder inequality, we see that

$$\mathrm{Var}(X_0(x)|X_0(y)) \geq 0, \quad \forall\, x, y \in I.$$

Taking (19) into account, it holds that for $x \neq y$,

$$\mathrm{Corr}(X_0(x), X_0(y))\sqrt{\mathbb{E}|X_0(x)|^2}\sqrt{\mathbb{E}|X_0(y)|^2} = \mathrm{Cov}(X_0(x), X_0(y)) < \sqrt{\mathbb{E}|X_0(x)|^2}\sqrt{\mathbb{E}|X_0(y)|^2},$$

which in combination with (23) indicates that $\mathrm{Var}(X_0(x)|X_0(y)) > 0$ for $x \neq y$. Noticing that $(x, y) \mapsto \mathrm{Var}(X_0(x)|X_0(y))$ is a continuous function on the bounded closed set I_2, we deduce that

$$\mathrm{Var}(X_0(x)|X_0(y)) \geq c_{10}, \quad \forall\, (x, y) \in I_2,$$

for some $c_{10} > 0$. By the boundedness of I, we conclude

$$\begin{aligned}\mathrm{Var}(X_0(x)|X_0(y)) &\geq \frac{c_{10}}{\max\limits_{x,y\in I}\sum_{j=1}^{m}\left|x_j - y_j\right|^{2H_j}}\sum_{j=1}^{m}\left|x_j - y_j\right|^{2H_j} \\ &\geq c_{11}\sum_{j=1}^{m}\left|x_j - y_j\right|^{2H_j}, \quad \forall\, (x, y) \in I_2,\end{aligned}$$

which completes the proof of (A.1) for *Case 2*.

Step 2: We prove the implication (C0), (C1) and (C2)$'$ $\Rightarrow$ (20), which is similar to that of [24, Theorem 5.10 and Theorem 5.11] (see also [9, Theorems 2.1 and 2.6]). We present a sketch proof here for the completeness.

Similar to [24, Theorem 5.10] or [9, Theorem 2.6], for the upper bound of (20), we need to verify

(i) $\inf_{x\in K}\mathbb{E}|X_0(x)|^2 > 0$, for any compact subset $K \subset I^\delta$ with some $\delta > 0$.

(ii) For any $\varepsilon > 0$ small enough,

$$\mathbb{E}\left[\int\limits_{R_j^\varepsilon}\int\limits_{R_j^\varepsilon}\exp\left(\frac{\|X(x) - X(y)\|}{\|x - y\|}\right)\mathrm{d}y\mathrm{d}x\right] \leq C\varepsilon^{2m},$$

where $R_j^\varepsilon = \prod_{l=1}^{m}[j_l\varepsilon, (j_l + 1)\varepsilon)$, $j = (j_1, \ldots, j_m) \in \mathbb{Z}^m$, and $R_j^\varepsilon \cap I \neq \emptyset$ (see [9, Theorems 2.6]).

Property (i) follows from (C0). Now we are devoted to the proof of property (ii). By the upper bound in (C1) with $H = (1, \ldots, 1)$, it holds for $i = 1, \ldots, d$,

$$\frac{|X_i(x) - X_i(y)|}{\|x - y\|} \leq \sqrt{c_3}\frac{|X_i(x) - X_i(y)|}{\sqrt{\mathbb{E}|X_i(x) - X_i(y)|^2}}.$$

Let $\Lambda_{x,y}$ be the covariance matrix of the Gaussian random vector $X(x) - X(y)$, i.e.,

$$\Lambda_{x,y} = \mathrm{diag}\left(\mathbb{E}|X_1(x) - X_1(y)|^2, \ldots, \mathbb{E}|X_d(x) - X_d(y)|^2\right),$$

because $X_i, i \in \{1, \ldots, d\}$ are mutually independent. The law of the random vector $Z := \Lambda_{x,y}^{-\frac{1}{2}}[X(x) - X(y)]$ is the normal distribution $\mathcal{N}(0, I_d)$. As a result,

$$\mathbb{E}\left[\int\limits_{R_j^\varepsilon}\int\limits_{R_j^\varepsilon}\exp\left(\frac{\|X(x) - X(y)\|}{\|x - y\|}\right)\mathrm{d}y\mathrm{d}x\right] \leq \int\limits_{R_j^\varepsilon}\int\limits_{R_j^\varepsilon}\mathbb{E}\left[e^{\sqrt{c_3}\|Z\|}\right]\mathrm{d}y\mathrm{d}x \leq C\varepsilon^{2m},$$

which proves property (ii).

Similar to [24, Theorem 5.11] or [9, Theorem 2.1], for the lower bound of (20), we need to verify

(iii) For any $x \in I$, the density function $z \mapsto p_{X(x)}(z)$ of $X(x)$ is continuous and bounded. Moreover, $p_{X(x)}(z) > 0$ for any z on a compact set of $\mathbb{R}^d$;

(iv) For any $x, y \in I$ with $x \neq y$, the density $p_{x,y}$ of the law of $(X(x), X(y))$ exists and satisfies: for any fixed $M_1 > 0$, there exist $\gamma, \alpha > 0$ such that $\frac{2}{\alpha}(\gamma - m) = d - Q$ with $Q = m$, and

$$p_{x,y}(z,\zeta) \leq \frac{C}{\|x-y\|^{\gamma}} \exp\left(-\frac{c\|z-\zeta\|^2}{\|x-y\|^{\alpha}}\right),$$

for any $z, \zeta \in [-M_1, M_1]^d$, where C, c are positive constants independent of x, y.

Let Λ_x be the covariance matrix of the Gaussian random vector $X(x)$, i.e.,

$$\Lambda_x = \operatorname{diag}\left(\mathbb{E}|X_1(x)|^2, \ldots, \mathbb{E}|X_d(x)|^2\right) = \operatorname{diag}\left(\mathbb{E}|X_0(x)|^2, \ldots, \mathbb{E}|X_0(x)|^2\right).$$

It follows from (C0) that Λ_x is positive definite, which together with the fact that $X(x)$ is Gaussian with mean zero and covariance matrix Λ_x, indicates property (iii).

It remains to show property (iv). For $x, y \in I$, denote $\sigma_{xy} := \operatorname{Cov}(X_0(x), X_0(y))$. Since $(X_0(x), X_0(y))$ has a Gaussian distribution with the covariance matrix

$$\Sigma_{x,y} := \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix},$$

by [14, Proposition 3.13], the conditional distribution of $X_0(x)$ given $X_0(y) = \zeta_0$ is

$$\mathcal{N}\left(\sigma_{xy}\sigma_{yy}^{-1}\zeta_0, \gamma_{xy}\right), \qquad \text{with } \gamma_{xy} := \sigma_{xx} - \sigma_{xy}\sigma_{yy}^{-1}\sigma_{xy}. \tag{A.5}$$

By (23) and (C2)$'$, we have that for $x \neq y$,

$$\gamma_{xy} = \operatorname{Var}(X_0(x)|X_0(y)) \geq c_4\|x-y\|^2 > 0. \tag{A.6}$$

Define the bilinear function on $\mathbb{R} \times \mathbb{R}$ by

$$(a,b)_* = a\gamma_{xy}^{-1}b, \quad \forall\, a, b \in \mathbb{R}.$$

Then the symmetry and positivity of γ_{xy}^{-1} ensure that $(\cdot,\cdot)_*$ is an inner product on $\mathbb{R}$, and the norm induced by which is denoted by $|\cdot|_*$. Applying the elementary inequality $-|a-b|_*^2 \leq -\frac{1}{2}|a|_*^2 + |b|_*^2$, $\forall\, a, b \in \mathbb{R}^2$, it follows from (A.5) that

$$\begin{aligned} q_{x|y}(z_0|\zeta_0) &= \frac{1}{\sqrt{2\pi\gamma_{xy}}} \exp\left(-\frac{1}{2}\left|z_0 - \sigma_{xy}\sigma_{yy}^{-1}\zeta_0\right|_*^2\right) \\ &\leq \frac{1}{\sqrt{2\pi\gamma_{xy}}} \exp\left(-\frac{1}{4}|z_0-\zeta_0|_*^2 + \frac{1}{2}\left|(1-\sigma_{xy}\sigma_{yy}^{-1})\zeta_0\right|_*^2\right), \end{aligned} \tag{A.7}$$

where $q_{x|y}(\cdot|\zeta_0)$ is the density of $\mathcal{N}(\sigma_{xy}\sigma_{yy}^{-1}\zeta_0, \gamma_{xy})$, i.e., $q_{x|y}(\cdot|\zeta_0)$ is the conditional density of $X_0(x)$ given $X_0(y) = \zeta_0$. Using (A.6),

$$\left|(1-\sigma_{xy}\sigma_{yy}^{-1})\zeta_0\right|_*^2 \leq c_4^{-1}\|x-y\|^{-2}\left|(1-\sigma_{xy}\sigma_{yy}^{-1})\zeta_0\right|^2.$$

Furthermore, by (C0), for any $\zeta_0 \in [-M_1, M_1]$,

$$\left|(1-\sigma_{xy}\sigma_{yy}^{-1})\zeta_0\right|^2 = \left|\sigma_{yy}^{-1}(\sigma_{xy}-\sigma_{yy})\zeta_0\right|^2 \leq c_1^{-2}M_1^2\left|\sigma_{xy}-\sigma_{yy}\right|^2.$$

Taking (C1) and (21) into account, we have

$$\begin{aligned} \left|\sigma_{xy}-\sigma_{yy}\right|^2 &= \left|\mathbb{E}[X_0(x)X_0(y)] - \mathbb{E}|X_0(y)|^2\right|^2 \\ &\leq \mathbb{E}|X_0(x)-X_0(y)|^2\,\mathbb{E}|X_0(y)|^2 \leq c_3c_7\|x-y\|^2, \end{aligned}$$

and thus $\left|(1-\sigma_{xy}\sigma_{yy}^{-1})\zeta_0\right|_*^2 \leq C$.

In view of (C1), (21), and the Hölder inequality,

$$0 \le \mathbb{E}|X_0(x) - X_0(y)|^2 - \left(\sqrt{\mathbb{E}|X_0(x)|^2} - \sqrt{\mathbb{E}|X_0(y)|^2}\right)^2 \le c_3\|x-y\|^2,$$
$$0 \le \left(\sqrt{\mathbb{E}|X_0(x)|^2} + \sqrt{\mathbb{E}|X_0(y)|^2}\right)^2 - \mathbb{E}|X_0(x) - X_0(y)|^2 \le 4c_7,$$

which along with (22) and (C0) give

$$\gamma_{xy} = \mathrm{Var}\,(X_0(x)|X_0(y)) \le \frac{4c_3c_7\|x-y\|^2}{4c_1} = \frac{c_3c_7\|x-y\|^2}{c_1}.$$

Thus, it holds that $|z_0-\zeta_0|_*^2 \ge \frac{c_1}{c_3c_7}\|x-y\|^{-2}|z_0-\zeta_0|^2 \ge C\|x-y\|^{-2}|z_0-\zeta_0|^2$.

Gathering the above estimates together leads to

$$q_{x|y}\,(z_0|\zeta_0) \le \frac{1}{\sqrt{2\pi c_4}\|x-y\|}\exp\left(-\frac{C|z_0-\zeta_0|^2}{4\|x-y\|^2}+\frac{1}{2}C\right) \le \frac{C}{\|x-y\|}\exp\left(-C\frac{|z_0-\zeta_0|^2}{\|x-y\|^2}\right),$$

for any $z_0, \zeta_0 \in [-M_1, M_1]^2$. Let $q_{X_0(y)}$ be the density of $X_0(y)$. Similar to property (iii), $q_{X_0(y)}(\zeta_0)$ is uniformly bounded for all $y \in I$ and $\zeta_0 \in [-M_1, M_1]$ since by (C1), $x \mapsto \mathbb{E}|X_0(x)|^2$ is continuous on I. Hence, we have that the density $q_{x,y}$ of $(X_0(x), X_0(y))$ satisfies for all $(z_0,\zeta_0) \in [-M_1, M_1]^2$,

$$q_{x,y}(z_0,\zeta_0) = q_{X_0(y)}(\zeta_0)q_{x|y}\,(z_0|\zeta_0) \le \frac{C}{\|x-y\|}\exp\left(-C\frac{|z_0-\zeta_0|^2}{\|x-y\|^2}\right). \tag{A.8}$$

Since X_i $i=1,\dots,d$, are independent copies of X_0, it follows from (A.8) that the density $p_{x,y}$ of

$$(X(x), X(y)) = (X_1(x),\dots,X_d(x),X_1(y),\dots,X_d(y))$$

satisfies for any $z=(z_1,\dots,z_d)$ and $\zeta=(\zeta_1,\dots,\zeta_d)$ in $[-M_1,M_1]^d$,

$$\begin{aligned} p_{x,y}(z,\zeta) &= \prod_{i=1}^{d} q_{x,y}(z_i,\zeta_i) \le \frac{C}{\|x-y\|^d}\exp\left(-C\sum_{i=1}^{d}\frac{|z_i-\zeta_i|^2}{\|x-y\|^2}\right)\\ &= \frac{C}{\|x-y\|^d}\exp\left(-C\frac{\|z-\zeta\|^2}{\|x-y\|^2}\right), \end{aligned}$$

which proves property (iv) with $\gamma = d$ and $\alpha = 2$.

A.3. Proof of Proposition 4.3

By (51), Itô's isometry gives

$$\mathbb{E}|\widetilde{Y}^0(t)|^2 = \int_0^t (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r \ge \int_0^{\frac{T_0}{2}} (1-\lambda\delta t)^{2[\frac{T+T_0/2}{\delta t}]}\mathrm{d}r > 0, \quad \forall\, t \in [\frac{T_0}{2}, T+\frac{T_0}{2}],$$

and for $s<t$,

$$\mathbb{E}|\widetilde{Y}^0(t)-\widetilde{Y}^0(s)|^2 = \int_s^t (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r + \int_0^s |(1-\lambda\delta t)^{[\frac{t-r}{\delta t}]} - (1-\lambda\delta t)^{[\frac{s-r}{\delta t}]}|^2\mathrm{d}r.$$

Since $1-\lambda\delta t \in (0,1)$, we have that for any $T_0 \le s < t \le T$,

$$(1-\lambda\delta t)^{2[\frac{T-T_0}{\delta t}]}(t-s) \le \int_s^t (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r \le (t-s)$$

and

$$\begin{aligned}\int_0^s |(1-\lambda\delta t)^{[\frac{t-r}{\delta t}]} - (1-\lambda\delta t)^{[\frac{s-r}{\delta t}]}|^2\mathrm{d}r &\le \int_0^s (1-\lambda\delta t)^{2[\frac{s-r}{\delta t}]} - (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r \\ &\le \int_0^s 1-(1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]-2[\frac{s-r}{\delta t}]}\mathrm{d}r =: J. \end{aligned} \tag{A.9}$$

The estimation of J is divided into two cases.

Case 1: $t-s \ge \delta t$.

It holds that $[\frac{t-r}{\delta t}] - [\frac{s-r}{\delta t}] \le \frac{t-r}{\delta t} - \frac{s-r}{\delta t} + 1 \le 2\frac{t-s}{\delta t}$, $\forall r \in (0,s)$. This implies that

$$\begin{aligned} J &\le \int_0^s 1-(1-\lambda\delta t)^{\frac{4(t-s)}{\delta t}}\mathrm{d}r = s\int_0^{\frac{4(t-s)}{\delta t}} -\frac{\mathrm{d}}{\mathrm{d}a}(1-\lambda\delta t)^a \mathrm{d}a \\ &= s\int_0^{\frac{4(t-s)}{\delta t}} (1-\lambda\delta t)^a \log\frac{1}{1-\lambda\delta t}\mathrm{d}a \le s\log\frac{1}{1-\lambda\delta t}\frac{4(t-s)}{\delta t} \le C(\delta t, T, \lambda)(t-s). \end{aligned}$$

Case 2: $0 < t-s < \delta t$.

Note that $s \in [t_m, t_{m+1})$ for some $0 \le m < T/\delta t$. Then

$$\begin{aligned} J &= \int_{t-\delta t}^{s} + \int_{s-\delta t}^{t-\delta t} + \int_{t-2\delta t}^{s-\delta t} + \cdots + \int_{s-m\delta t}^{t-m\delta t} + \int_{\max\{0,t-(m+1)\delta t\}}^{s-m\delta t} \\ &\quad + \int_0^{\max\{0,t-(m+1)\delta t\}} 1-(1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]-2[\frac{s-r}{\delta t}]}\mathrm{d}r. \end{aligned}$$

Notice that for $i \in \{0,1,\dots,m\}$, we have $[\frac{t-r}{\delta t}] = [\frac{s-r}{\delta t}] = i$, $\forall r \in (t-(i+1)\delta t, s-i\delta t)$. Therefore,

$$\begin{aligned} J &= \sum_{i=1}^m \int_{s-i\delta t}^{t-i\delta t} + \int_0^{\max\{0,t-(m+1)\delta t\}} 1-(1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]-2[\frac{s-r}{\delta t}]}\mathrm{d}r \\ &\le (m+1)(t-s) \le \left(\frac{T}{\delta t}+1\right)(t-s), \end{aligned}$$

where $t-(m+1)\delta t < t-s$ is used. Combining *Case 1* and *Case 2* yields $J \le C(T, \delta t, \lambda)(t-s)$.

Similar to the proof of (34), we also have for $t \ne s$,

$$\mathbb{E}|\tilde{Y}_0(t)|^2\,\mathbb{E}|\tilde{Y}_0(s)|^2 - |\mathbb{E}[\tilde{Y}_0(t)\tilde{Y}_0(s)]|^2 > 0.$$

In order to apply Proposition 3.1, it suffices to prove that $t \mapsto \mathbb{E}|\widetilde{Y}^0(t)|^2$ is Hölder continuous with exponent $H_0 > 1/2$. Indeed, it follows immediately from the above estimates on (A.9) that

$$\begin{aligned}\left|\mathbb{E}|\widetilde{Y}^0(t)|^2-\mathbb{E}|\widetilde{Y}^0(s)|^2\right| &= \left|\int_s^t (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r - \int_0^s (1-\lambda\delta t)^{2[\frac{s-r}{\delta t}]} - (1-\lambda\delta t)^{2[\frac{t-r}{\delta t}]}\mathrm{d}r\right| \\ &\leq (t-s)+J \\ &\leq C(T,\delta t,\lambda)(t-s).\end{aligned}$$

Finally, we complete the proof by applying Proposition 3.1 with $I=[T_0,T]$ and $\delta=T_0/2$. □

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